

Branching random walks in random environment and super-Brownian motion in random environment

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Abstract

We focus on the existence and characterization of the limit for a certain critical branching random walks in time-space random environment in one dimension which was introduced in [2]. Each particle performs simple random walk on \mathbb{Z} and branching mechanism depends on the time-space site. The weak limit points of this measure valued processes are characterized as solutions of the non-trivial martingale problem and called super-Brownian motions in random environment in [17]. Moreover, we will show the weak uniqueness of the solutions with some initial condition.

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We denote by (Ω, \mathcal{F}, P) a probability space. Let $\mathbb{N} = \{0, 1, 2, \dots\}$, $\mathbb{N}^* = \{1, 2, 3, \dots\}$, and $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$.

1 Introduction

Super-Brownian motion(SBM) is a measure valued process which was introduced by Dawson and Watanabe independently[4, 27] and is obtained as the limit of (asymptotically) critical branching Brownian motions (or branching random walks). There are many books for introduction of super-Brownian motion [5, 9] and dealing with several aspects of it [7, 8, 13, 23]. Also, super-Brownian motion appears in physics and population genetics.

An example of the construction is the following, where we always treat Euclidean space as the space, \mathbb{R}^d in this paper.

We assume that at time 0, there are N particles in \mathbb{Z}^d as the 0-th generation particle. Each of N particles chooses independently of each others a nearest neighbor site uniformly, moves there at time 1, and then each particle independently of each others either dies or split into two particles with probability 1/2 (1st generation). The newly produced particles in n -th generation perform in the same manner, that is each of them chooses independently of each others

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a nearest neighbor site uniformly, moves there at time $n + 1$, and then each particle independently of each others either dies or split into 2 particles with probability $1/2$.

Let $X_t^{(N)}(\cdot)$ be the measure-valued Markov processes defined by

$$X_t^{(N)}(B) = \frac{\# \left\{ \text{particles in } B\sqrt{N} \text{ at } \lfloor tN \rfloor\text{-th generation at time } tN \right\}}{N},$$

where $B \in \mathcal{B}(\mathbb{R})$ are Borel sets in \mathbb{R}^d and $B\sqrt{N} = \{x = y\sqrt{N} \text{ for } y \in B\}$. Then, under some conditions, they converge as $N \rightarrow \infty$ to a measure-valued processes, *super-Brownian motion*. In particular, the limit $X_t(\phi)$ is characterized as the unique solution of the martingale problem:

$$\left\{ \begin{array}{l} \text{For all } \phi \in \mathcal{D}(\Delta), \\ Z_t(\phi) := X_t(\phi) - X_0(\phi) - \int_0^t \frac{1}{2d} X_s(\Delta\phi) ds \\ \text{is an } \mathcal{F}_t^X\text{-continuous square-integrable martingale} \\ Z_0(\phi) = 0 \text{ and } \langle Z(\phi) \rangle_t = \int_0^t X_s(\phi^2) ds, \end{array} \right. \quad (1.1)$$

where $\nu(\phi) = \int \phi d\nu$ for any measure ν .

It is a well-known fact that one-dimensional super-Brownian motion is related to stochastic heat equation([12, 24]). When $d = 1$, super-Brownian motion $X_t(dx)$ is almost surely absolutely continuous with respect to Lebesgue measure and its density $u(t, x)$ satisfies the following stochastic heat equation:

$$\frac{\partial}{\partial t} u = \frac{1}{2} \Delta u + \sqrt{u} \dot{W}(t, x),$$

where $\dot{W}(t, x)$ is space-time white noise. On the other hand, for $d \geq 2$, $X_t(\cdot)$ is almost singular with respect to Lebesgue measure.([6, 14, 21, 22])

In this paper, we consider super-Brownian motion in random environment, which are introduced in [17]. Mytnik showed the existence and uniqueness of the scaling limit $X_t(\cdot)$ for a certain critical branching diffusion in random environment with some conditions. It is characterized as the unique solution of the martingale problem:

$$\left\{ \begin{array}{l} \text{For all } \phi \in \mathcal{D}(\Delta), \\ Z_t(\phi) := X_t(\phi) - X_0(\phi) - \int_0^t \frac{1}{2} X_s(\Delta\phi) ds \\ \text{is an } \mathcal{F}_t^X\text{-continuous square-integrable martingale and} \\ \langle Z(\phi) \rangle_t = \int_0^t X_s(\phi^2) ds + \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} g(x, y) \phi(x) \phi(y) X_s(dx) X_s(dy) ds, \end{array} \right. \quad (1.2)$$

where $g(\cdot, \cdot)$ is bounded continuous function in a certain class. In this paper, we construct a super-Brownian motion in random environment as a limit point of scaled branching random walks in random environment, which is a solution of (1.2) for the case where $g(x, y)$ is replaced by $\delta_{x, y}$. The definition of such martingale problem is formal. The rigorous definition will be given later.

2 Branching random walks in random environment

Before giving the system of the branching random walks in random environment, we introduce Ulam-Harris tree \mathcal{T} for labeling the particles. We set $T_k = (\mathbb{N}^*)^{k+1}$ for $k \geq 1$. Then, Ulam-Harris tree \mathcal{T} is defined by $\mathcal{T} = \bigcup_{k \geq 0} T_k$.

We will give a name to each particle by using elements of \mathcal{T} .

- i) When there are M particles at the 0-th generation, we label them as $1, 2, \dots, M \in T_0$.
- ii) If the n -th generation particle $\mathbf{x} = (x_0, \dots, x_n) \in T_n$ gives birth to $k_{\mathbf{x}}$ particles, then we name them as $(x_0, \dots, x_n, 1), \dots, (x_0, \dots, x_n, k_{\mathbf{x}}) \in T_{n+1}$.

Thus, every particle in the branching systems has its own name in \mathcal{T} . We define $|\mathbf{x}|$ by its generation, that is if \mathbf{x} is an element of T_k , then $|\mathbf{x}| = k$. For convenience, we denote by $|\mathbf{x} \wedge \mathbf{y}|$ the generation of the closest common ancestor of \mathbf{x} and \mathbf{y} . If \mathbf{x} and \mathbf{y} have no common ancestor, then we define $|\mathbf{x} \wedge \mathbf{y}| = -\infty$. Also, we denote by \mathbf{y}/\mathbf{x} the last digit of \mathbf{y} when \mathbf{y} is a child of \mathbf{x} , that is

$$\mathbf{y}/\mathbf{x} = \begin{cases} k_{\mathbf{y}}, & \text{if } \mathbf{x} = (x_0, \dots, x_n) \in T_n, \\ & \mathbf{y} = (x_0, \dots, x_n, k_{\mathbf{y}}) \in T_{n+1}, \text{ for some } n \in \mathbb{N}, \\ \infty, & \text{otherwise.} \end{cases}$$

Now, we give the definition of branching random walks in random environment. In our case, particle move on \mathbb{Z} and the process evolves by the following rule:

- i) The initial particles locate at site $\{x_i \in 2\mathbb{Z} : i = 1, \dots, M_N\}$.
- ii) Each particle located at site x at time n chooses a nearest neighbor site independently of each others with probability $\frac{1}{2}$ and moves there at time $n+1$. Simultaneously, it is replaced by k -children with probability $q_{n,x}^{(N)}(k)$ independently of each others,

where $\left\{ \left\{ q_{n,x}^{(N)}(k) \right\}_{k=0}^{\infty} : (n, x) \in \mathbb{N} \times \mathbb{Z} \right\}$ are the offspring distributions assigned to each time-space site (n, x) which are i.i.d. in (n, x) . We denote by $B_n^{(N)}$ and by $B_{n,x}^{(N)}$ the total number of particles at time n and the local number of particles at site x at time n . Also, we denote by $m_{n,x}^{(N,p)}$ the p -th moment of offsprings for offspring distribution $\{q_{n,x}^{(N)}(k)\}$, that is

$$m_{n,x}^{(N,p)} = \sum_{k=0}^{\infty} k^p q_{n,x}^{(N)}(k).$$

This model is called branching random walks in random environment (BRWRE) whose properties as measure valued processes is for “supercritical” case are studied well [10, 11]. Also, the continuous counterpart, branching Brownian motions in random environment is introduced by Shiozawa[25, 26]. We know that the normalized random measure weakly converges to Gaussian measure

in probability in one phase, whereas the localization has occurred in the other phase.

In this paper, we focus on the scaled measure valued processes $X_t^{(N)}$ associated to this branching random walks:

$$X_0^{(N)} = \frac{1}{N} \sum_{i=0}^{M_N} \delta_{x_i/N^{\frac{1}{2}}},$$

and

$$X_t^{(N)} = \frac{1}{N} \sum_{i=1}^{B_{tN}^{(N)}} \delta_{x_i(t)/N^{\frac{1}{2}}}, \quad \text{for } t = \frac{1}{N}, \dots, \frac{\lfloor KN \rfloor}{N} \text{ for each } K > 0,$$

where $x_i(t)$ is the position of the i -th particle at tN -th generation. We remark that if we identify $B_{tN,x}^{(N)}$ as the measure $B_{tN,x}^{(N)} \delta_x$, then $X_t^{(N)}$ is represented as

$$X_t^{(N)} = \frac{1}{N} \sum_{x \in \mathbb{Z}} B_{tN,x}^{(N)} \delta_{x/N^{\frac{1}{2}}} \quad \text{for } t = \frac{1}{N}, \dots, \frac{\lfloor KN \rfloor}{N}.$$

Let $\mathcal{M}_F(\mathbb{R})$ be the set of the finite Borel measures on \mathbb{R} . For convenience, we extend this model to the càdlàg paths in $\mathcal{M}_F(\mathbb{R})$ by

$$X_t^{(N)} = \frac{1}{N} \sum_{x \in \mathbb{Z}} B_{\underline{t}N,x}^{(N)} \delta_{x/N^{\frac{1}{2}}}, \quad \text{for } \underline{t} \leq t < \underline{t} + \frac{1}{N},$$

where we define \underline{t} for t and N by some positive number $\frac{i}{N}$ for $i \in \mathbb{N}$ satisfying $\frac{i}{N} \leq t < \frac{i+1}{N}$. Then, $X_t^{(N)} \in \mathcal{M}_F(\mathbb{R})$ for each $t \in [0, K]$. Let $\phi \in \mathcal{B}_b(\mathbb{R})$, where $\mathcal{B}_b(\mathbb{R})$ is the set of the bounded Borel measurable functions on \mathbb{R} . We denote the product of $\nu \in \mathcal{M}_F(\mathbb{R})$ and $\phi \in \mathcal{B}_b(\mathbb{R})$ by $\nu(\phi)$, that is

$$\nu(\phi) = \int_{\mathbb{R}} \phi(x) \nu(dx).$$

To describe the main theorem, we give the following assumption on the environment:

Assumption A

$$\begin{aligned} E[m_{0,0}^{(N,1)}] &= E \left[\sum_{i=0}^{\infty} k q_{n,x}^{(N)} \right] = 1, \quad \lim_{N \rightarrow \infty} E \left[m_{0,0}^{(N,2)} - 1 \right] = \gamma > 0, \\ \sup_{N \geq 1} E \left[m_{0,0}^{(N,4)} \right] &< \infty, \quad \lim_{N \rightarrow \infty} N^{\frac{1}{2}} E \left[(m_{0,0}^{(N,1)} - 1)^2 \right] = \beta^2, \\ \sup_{N \geq 1} N^{\frac{1}{2}} E \left[(m_{0,0}^{(N,1)} - 1)^4 \right] &< \infty. \end{aligned}$$

Example: The simplest example satisfying Assumption A is the case where $q_{n,x}^{(N)}(0) = \frac{1}{2} - \frac{\beta \xi(n,x)}{2N^{\frac{1}{4}}}$, $q_{n,x}^{(N)}(2) = \frac{1}{2} + \frac{\beta \xi(n,x)}{2N^{\frac{1}{4}}}$ for i.i.d. random variables $\{\xi(n,x) : (n,x) \in \mathbb{N} \times \mathbb{Z}\}$ such that $P(\xi(n,x) = 1) = P(\xi(n,x) = -1) = \frac{1}{2}$.

Theorem 2.1. *We suppose that $X_0^{(N)}(\cdot) \Rightarrow X_0(\cdot)$ in $\mathcal{M}_F(\mathbb{R})$ and Assumption A. Then, the sequence of measure valued processes $\{X^{(N)} : N \in \mathbb{N}\}$ is tight and its weak limit point weakly is a continuous measure valued process $X. \in C([0, \infty), \mathcal{M}_F(\mathbb{R}))$. Moreover, for any $t > 0$, any limit point $X_t(dx)$ is almost surely absolutely continuous with respect to Lebesgue measure and its density $u(t, x)$ is a solution of the following martingale problem:*

$$\left\{ \begin{array}{l} \text{For all } \phi \in \mathcal{D}(\Delta), \\ Z_t(\phi) = \int_{\mathbb{R}} \phi(x) u(t, x) dx - \int_{\mathbb{R}} \phi(x) X_0(dx) - \frac{1}{2} \int_0^t \int_{\mathbb{R}} \Delta \phi(x) u(s, x) dx ds \\ \text{is an } \mathcal{F}_t^X\text{-continuous square-integrable martingale and} \\ \langle Z(\phi) \rangle_t = \int_0^t \int_{\mathbb{R}} \phi^2(x) (\gamma u(s, x) + 2\beta^2 u(s, x)^2) dx ds. \end{array} \right. \quad (2.1)$$

Remark: We found from Assumption A that the fluctuation of the environment is mainly given by $(m_{n,x}^{(N,1)} - 1)$ and scaling factor is $N^{-\frac{1}{4}}$. (It appears clearly in the Example beyond Assumption A.) This scaling factor is different from $N^{-\frac{1}{2}}$, the one in [17]. When the scaling factor is $N^{-\frac{1}{2}}$, the limit is the usual super-Brownian motion (1.1).

We roughly discuss how the scaling factor in our model is determined. For simplicity, we consider the model for the case where the environment is the one given in Example.

We scale the space by $N^{-\frac{1}{2}}$. Then, the summation of the fluctuation of the first moment of offsprings in the segment $\{k\} \times [x, y]$ is $\sum_{z \in [xN^{1/2}, yN^{1/2}]} \frac{\beta \xi(k, z)}{N^{\frac{1}{4}}}$.

Since it is the summation of i.i.d. random variables of $\frac{(y-x)N^{\frac{1}{2}}}{2}$, the central limit theorem holds and it weakly converges to a Gaussian random variable with distribution $N(0, \frac{\beta^2(y-x)}{2})$. Similar argument holds for random variables other than Bernoulli random variables.

Remark: The martingale problem (2.1) is the rigorous and general definition of the martingale problem when $g(x, y)$ is replaced by δ_{x-y} in (1.2). Also, the theorem implies the existence of the solution of the stochastic heat equation

$$\frac{\partial}{\partial t} u = \frac{1}{2} \Delta u + \sqrt{\gamma u + 2\beta^2 u^2} \dot{W}, \quad (2.2)$$

and $\lim_{t \rightarrow +0} u(t, x) dx = X_0(dx)$, where \dot{W} is time-space white noise. In [16], the existence of solutions for general SPDE containing (2.2) when the initial measure $X_0(dx)$ has a continuous density with rapidly decreasing at infinity.

Also, one of our interest is the uniqueness of solutions of (2.2). There are a lot of papers on uniqueness of the stochastic heat equation $\frac{\partial}{\partial t} u = \frac{1}{2} \Delta u + |u|^\gamma \dot{W}$. It is known that weak uniqueness holds for $\frac{1}{2} \leq \gamma \leq 1$ in [18] and pathwise uniqueness holds for $\frac{3}{4} < \gamma \leq 1$ in [19]. However, pathwise uniqueness fails when solutions are allowed to take negative values for $\gamma < \frac{3}{4}$ in [15].

In this paper, we will prove the uniqueness under some strong initial condition. We denote by $C_{rap}^+(\mathbb{R})$ the set of the rapidly decreasing continuous

functions,

$$C_{rap}^+(\mathbb{R}) = \left\{ g \in C_b^+(\mathbb{R}) : |g|_p = \sup_x e^{p|x|} g(x) < \infty, \text{ for all } p > 0 \right\},$$

where $C_b^+(\mathbb{R})$ is the set of bounded continuous functions on \mathbb{R} . We can consider $C_{rap}^+(\mathbb{R})$ as a subset of $\mathcal{M}_F(\mathbb{R})$ by $\phi(x) \mapsto \phi(x)dx$.

We will prove the following theorem.

Theorem 2.2. *Assume that $X_0(dx) = \psi(x)dx$ for $\psi \in C_{rap}^+(\mathbb{R})$. Let $a, b > 0$. Then, any two solutions for the stochastic heat equation,*

$$\frac{\partial}{\partial t} u = \frac{1}{2} \Delta u + \sqrt{au + bu^2} \dot{W}(t, x), \quad u(0, x) = \psi(x) \quad (2.3)$$

have the same finite dimensional distributions. In particular, the solution of (2.3) is unique.

3 Proof of Theorem 2.1

In this section, we will give a proof of Theorem 2.1. The proof is divided into two steps:

- i) Tightness.
- ii) Identification of the limit point process.

In this section, we consider the following setting for simplicity.

Assumption B: The number of initial particles is N and all of them locates at the origin at time 0. Also, $q_{n,x}^{(N)}(0) = \frac{1}{2} - \frac{\beta\xi(n,x)}{2N^{\frac{1}{4}}}$, $q_{n,x}^{(N)}(2) = \frac{1}{2} + \frac{\beta\xi(n,x)}{2N^{\frac{1}{4}}}$ for i.i.d. random variables $\{\xi(n,x) : (n,x) \in \mathbb{N} \times \mathbb{Z}\}$ such that $P(\xi(n,x) = 1) = P(\xi(n,x) = -1) = \frac{1}{2}$.

To consider the general model, it is almost enough to replace $\frac{\beta\xi(n,x)}{N^{\frac{1}{4}}}$ by $m_{n,x}^{(N,1)} - 1$. We sometimes need to consider $\{q_{n,x}^{(N)}(k)\}_{k \geq 0} : (n,x) \in \mathbb{N} \times \mathbb{Z}\}$. Especially, γ appears in the same situation as the construction of the usual super-Brownian motion, so the reader will not to have any difficulties.

Before staring the proof, we will look at the $X_t^{(N)}(\phi)$. Since $X_t^{(N)}$ are constant in $t \in [\underline{t}, \underline{t} + \frac{1}{N})$, it is enough to see the difference between $X_{\underline{t}}^{(N)}$ and $X_{\underline{t} + \frac{1}{N}}^{(N)}$;

$$X_{\underline{t} + \frac{1}{N}}^{(N)}(\phi) - X_{\underline{t}}^{(N)}(\phi) = \frac{1}{N} \sum_{\mathbf{x} \sim \underline{t}} \left(\phi \left(\frac{Y_{\underline{t}N+1}^{\mathbf{x}}}{N^{\frac{1}{2}}} \right) V^{\mathbf{x}} - \phi \left(\frac{Y_{\underline{t}N}^{\mathbf{x}}}{N^{\frac{1}{2}}} \right) \right),$$

where $\mathbf{x} \sim \underline{t}$ means that the particle \mathbf{x} is the $\underline{t}N$ -th generation, $Y_{\underline{t}N}^{\mathbf{x}}$ is the position of the particle \mathbf{x} at time $\underline{t}N$ for $\mathbf{x} \sim \underline{t}$, $V^{\mathbf{x}}$ is the number of children of \mathbf{x} and for simplicity, we omit N . We remark that $Y_{\underline{t}N+1}^{\mathbf{x}} = Y_{\underline{t}N+1}^{\mathbf{y}}$ for \mathbf{y} which is a child of \mathbf{x} .

Also, we divide this summation into four parts:

$$\begin{aligned}
(LHS) &= \frac{1}{N} \sum_{\mathbf{x} \sim \underline{t}} \phi \left(\frac{Y_{\underline{t}N+1}^{\mathbf{x}}}{N^{\frac{1}{2}}} \right) \left(V^{\mathbf{x}} - 1 - \frac{\beta \xi(\underline{t}N, Y_{\underline{t}N}^{\mathbf{x}})}{N^{\frac{1}{4}}} \right) \\
&\quad + \frac{1}{N} \sum_{\mathbf{x} \sim \underline{t}} \phi \left(\frac{Y_{\underline{t}N+1}^{\mathbf{x}}}{N^{\frac{1}{2}}} \right) \frac{\beta \xi(\underline{t}N, Y_{\underline{t}N}^{\mathbf{x}})}{N^{\frac{1}{4}}} \\
&\quad + \frac{1}{N} \sum_{\mathbf{x} \sim \underline{t}} \left(\phi \left(\frac{Y_{\underline{t}N+1}^{\mathbf{x}}}{N^{\frac{1}{2}}} \right) - \phi \left(\frac{Y_{\underline{t}N}^{\mathbf{x}}}{N^{\frac{1}{2}}} \right) - \frac{\phi \left(\frac{Y_{\underline{t}N+1}^{\mathbf{x}}}{N^{\frac{1}{2}}} \right) + \phi \left(\frac{Y_{\underline{t}N}^{\mathbf{x}}}{N^{\frac{1}{2}}} \right) - 2\phi \left(\frac{Y_{\underline{t}N}^{\mathbf{x}}}{N^{\frac{1}{2}}} \right)}{2} \right) \\
&\quad + \frac{1}{N} \sum_{\mathbf{x} \sim \underline{t}} \frac{\phi \left(\frac{Y_{\underline{t}N+1}^{\mathbf{x}}}{N^{\frac{1}{2}}} \right) + \phi \left(\frac{Y_{\underline{t}N}^{\mathbf{x}}}{N^{\frac{1}{2}}} \right) - 2\phi \left(\frac{Y_{\underline{t}N}^{\mathbf{x}}}{N^{\frac{1}{2}}} \right)}{2} \\
&= \Delta M_{\underline{t}}^{(b,N)}(\phi) + \Delta M_{\underline{t}}^{(e,N)}(\phi) + \Delta M_{\underline{t}}^{(s,N)}(\phi) \\
&\quad + \frac{1}{N} \sum_{\mathbf{x} \sim \underline{t}} \frac{\phi \left(\frac{Y_{\underline{t}N+1}^{\mathbf{x}}}{N^{\frac{1}{2}}} \right) + \phi \left(\frac{Y_{\underline{t}N}^{\mathbf{x}}}{N^{\frac{1}{2}}} \right) - 2\phi \left(\frac{Y_{\underline{t}N}^{\mathbf{x}}}{N^{\frac{1}{2}}} \right)}{2}.
\end{aligned}$$

Thus, we have that for $0 \leq \underline{t} \leq t < \underline{t} + \frac{1}{N}$

$$X_t^{(N)}(\phi) - X_0^{(N)}(\phi) = \left(M_t^{(b,N)}(\phi) + M_t^{(e,N)}(\phi) + M_t^{(s,N)}(\phi) \right) + \int_0^t X_s^{(N)}(A^N \phi) ds, \quad (3.1)$$

where

$$\begin{aligned}
M_t^{(b,N)}(\phi) &= \frac{1}{N} \sum_{\underline{s} < t} \sum_{\mathbf{x} \sim \underline{s}} \phi \left(\frac{Y_{\underline{s}N+1}^{\mathbf{x}}}{N^{\frac{1}{2}}} \right) \left(V^{\mathbf{x}} - 1 - \frac{\beta \xi(\underline{s}N, Y_{\underline{s}N}^{\mathbf{x}})}{N^{\frac{1}{4}}} \right), \\
M_t^{(e,N)}(\phi) &= \frac{1}{N} \sum_{\underline{s} < t} \sum_{\mathbf{x} \sim \underline{s}} \phi \left(\frac{Y_{\underline{s}N+1}^{\mathbf{x}}}{N^{\frac{1}{2}}} \right) \frac{\beta \xi(\underline{s}N, Y_{\underline{s}N}^{\mathbf{x}})}{N^{\frac{1}{4}}}, \\
M_t^{(s,N)}(\phi) &= \frac{1}{N} \sum_{\underline{s} < t} \sum_{\mathbf{x} \sim \underline{s}} \left(\phi \left(\frac{Y_{\underline{s}N+1}^{\mathbf{x}}}{N^{\frac{1}{2}}} \right) - \phi \left(\frac{Y_{\underline{s}N}^{\mathbf{x}}}{N^{\frac{1}{2}}} \right) - \frac{\phi \left(\frac{Y_{\underline{s}N+1}^{\mathbf{x}}}{N^{\frac{1}{2}}} \right) + \phi \left(\frac{Y_{\underline{s}N}^{\mathbf{x}}}{N^{\frac{1}{2}}} \right) - 2\phi \left(\frac{Y_{\underline{s}N}^{\mathbf{x}}}{N^{\frac{1}{2}}} \right)}{2} \right),
\end{aligned}$$

and $A^N : \mathcal{B}_b(\mathbb{R}) \rightarrow \mathcal{B}_b(\mathbb{R})$ is the following operator;

$$A^N \phi(x) = \frac{\phi \left(x + \frac{1}{N^{\frac{1}{2}}} \right) + \phi \left(x - \frac{1}{N^{\frac{1}{2}}} \right) - 2\phi(x)}{\frac{2}{N}}.$$

Actually, we have that

$$\int_0^t X_s^{(N)}(A^N \phi) ds = \sum_{\underline{s} < t} \sum_{\mathbf{x} \sim \underline{s}} \frac{1}{N} A^N \phi \left(\frac{Y_{\underline{s}N}^{\mathbf{x}}}{N^{\frac{1}{2}}} \right).$$

Also, we remark that $M_{\underline{t}}^{(b,N)}(\phi)$, $M_{\underline{t}}^{(e,N)}(\phi)$, and $M_{\underline{t}}^{(s,N)}(\phi)$ are $\mathcal{F}_{\underline{t}N}^{(N)}$ -martingales, where $\mathcal{F}_n^{(N)}$ is the σ -algebra

$$\sigma(V^{\mathbf{x}}, Y_{k+1}^{\mathbf{x}}, \xi(k, x) : |\mathbf{x}| \leq n-1, k \leq n-1, x \in \mathbb{Z}),$$

where $\mathcal{F}_0^{(N)} = \{\emptyset, \Omega\}$. Indeed, since $Y_{n+1}^{\mathbf{x}}$ are independent of $V^{\mathbf{x}}$ and $\xi(n, x)$,

$$\begin{aligned} & E \left[M_{\underline{t}}^{(b,N)}(\phi) - M_{\underline{t}-\frac{1}{N}}^{(b,N)}(\phi) \middle| \mathcal{F}_{\underline{t}N-1}^{(N)} \right] \\ &= \frac{1}{N} \sum_{\mathbf{x} \sim \underline{t}-\frac{1}{N}} E \left[\phi \left(\frac{Y_{\underline{t}N}^{\mathbf{x}}}{N^{\frac{1}{2}}} \right) \middle| \mathcal{F}_{\underline{t}N-1}^{(N)} \right] E \left[V^{\mathbf{x}} - 1 - \frac{\beta \xi(\underline{t}N-1, Y_{\underline{t}N-1}^{\mathbf{x}})}{N^{\frac{1}{4}}} \middle| \mathcal{F}_{\underline{t}N-1}^{(N)} \right] \\ &= 0, \\ & E \left[M_{\underline{t}}^{(e,N)}(\phi) - M_{\underline{t}-\frac{1}{N}}^{(e,N)}(\phi) \middle| \mathcal{F}_{\underline{t}N-1}^{(N)} \right] \\ &= \frac{1}{N} \sum_{\mathbf{x} \sim \underline{t}-\frac{1}{N}} E \left[\phi \left(\frac{Y_{\underline{t}N}^{\mathbf{x}}}{N^{\frac{1}{2}}} \right) \middle| \mathcal{F}_{\underline{t}N-1}^{(N)} \right] E \left[\frac{\beta \xi(\underline{t}N-1, Y_{\underline{t}N-1}^{\mathbf{x}})}{N^{\frac{1}{4}}} \middle| \mathcal{F}_{\underline{t}N-1}^{(N)} \right] \\ &= 0, \end{aligned}$$

and

$$E \left[M_{\underline{t}}^{(s,N)}(\phi) - M_{\underline{t}-\frac{1}{N}}^{(s,N)}(\phi) \middle| \mathcal{F}_{\underline{t}N-1}^{(N)} \right] = 0,$$

almost surely.

Moreover, the decomposition (3.1) is very useful since the martingales $M_{\underline{t}}^{(i,N)}(\phi)$ $i = b, e, s$ are orthogonal to each others. Indeed, we have that

$$\begin{aligned} & E \left[\left(\Delta M_{\underline{t}}^{(b,N)}(\phi) \right) \left(\Delta M_{\underline{t}}^{(e,N)}(\phi) \right) \middle| \mathcal{F}_{\underline{t}N-1}^{(N)} \right] \\ &= \frac{1}{N^2} \sum_{\mathbf{x}, \mathbf{x}' \sim \underline{t}-\frac{1}{N}} \left(E \left[\phi \left(\frac{Y_{\underline{t}N}^{\mathbf{x}}}{N^{\frac{1}{2}}} \right) \phi \left(\frac{Y_{\underline{t}N}^{\mathbf{x}'}}{N^{\frac{1}{2}}} \right) \middle| \mathcal{F}_{\underline{t}N-1}^{(N)} \right] \right. \\ &\quad \times E \left[E \left[\left(V^{\mathbf{x}} - 1 - \frac{\beta \xi(\underline{t}N-1, Y_{\underline{t}N-1}^{\mathbf{x}})}{N^{\frac{1}{4}}} \right) \middle| \mathcal{G}_{\underline{t}N-1}^{(N)} \right] \frac{\beta \xi(\underline{t}N-1, Y_{\underline{t}N-1}^{\mathbf{x}'})}{N^{\frac{1}{4}}} \middle| \mathcal{F}_{\underline{t}N-1}^{(N)} \right] \Bigg) \\ &= 0, \end{aligned}$$

where $\mathcal{G}_n^{(N)} = \mathcal{F}_n^{(N)} \vee \sigma(\xi(n, x) : x \in \mathbb{Z})$ almost surely. Also, we can obtain by the similar argument that $E \left[\left(\Delta M_{\underline{t}}^{(b,N)}(\phi) \right) \left(\Delta M_{\underline{t}}^{(s,N)}(\phi) \right) \middle| \mathcal{F}_{\underline{t}N-1}^{(N)} \right] = E \left[\left(\Delta M_{\underline{t}}^{(s,N)}(\phi) \right) \left(\Delta M_{\underline{t}}^{(e,N)}(\phi) \right) \middle| \mathcal{F}_{\underline{t}N-1}^{(N)} \right] = 0$ almost surely.

3.1 Tightness

In this subsection, we will prove the following lemma.

Lemma 3.1. *The sequence $\{X^{(N)}\}$ is tight in $D([0, \infty), \mathcal{M}_F(\mathbb{R}))$, and each limit process is continuous.*

To prove it, we will use the following theorem which reduces the problem to the tightness of real-valued process [23, Theorem II. 4. 1].

Theorem 3.2. *Assume that E is a Polish space. Let D_0 be a separating class of $C_b(E)$ containing 1. A sequence of càdlàg $\mathcal{M}_F(E)$ -valued processes $\{X^{(N)} : N \in \mathbb{N}\}$ is C -relatively compact in $D([0, \infty), \mathcal{M}_F(E))$ if and only if*

(i) *for every $\varepsilon, T > 0$, there is a compact set $K_{T,\varepsilon}$ in E such that*

$$\sup_N P \left(\sup_{t \leq T} X_t^{(N)} (K_{T,\varepsilon}^c) > \varepsilon \right) < \varepsilon,$$

(ii) *and for all $\phi \in D_0$, $\{X^{(N)}(\phi) : N \in \mathbb{N}\}$ is C -relatively compact in $D([0, \infty), \mathbb{R})$.*

Assumption: We choose $C_b^2(\mathbb{R})$ as D_0 , where $C_b^2(\mathbb{R})$ is the set of bounded continuous function on \mathbb{R} with bounded derivatives of order 1 and 2.

Hereafter, we will check the conditions (i) and (ii) of Theorem 3.2 for our case. In the beginning, we give the proof of (ii) by using the following lemmas:

Lemma 3.3. *For $\phi \in C_b^2(\mathbb{R})$, $\sup_{t \leq K} |M_t^{(s,N)}(\phi)| \xrightarrow{L^2} 0$ as $N \rightarrow \infty$ for all $K > 0$.*

Lemma 3.4. (See [23, Lemma II 4.5].) *Let $(M_{\underline{t}}^{(N)}, \overline{\mathcal{F}}_{\underline{t}}^N)$ be discrete time martingales with $M_0^{(N)} = 0$.*

Let $\langle M^{(N)} \rangle_{\underline{t}} = \sum_{0 \leq \underline{s} < \underline{t}} E \left[\left(M_{\underline{s}+1/N}^{(N)} - M_{\underline{s}}^{(N)} \right)^2 \middle| \overline{\mathcal{F}}_{\underline{s}}^N \right]$, and we extend $M^{(N)}$ and $\langle M^{(N)} \rangle$ to $[0, \infty)$ as right continuous step functions.

(i) *If $\{\langle M^{(N)} \rangle : N \in \mathbb{N}\}$ is C -relatively compact in $D([0, \infty), \mathbb{R})$ and*

$$\sup_{0 \leq \underline{t} \leq K} |M_{\underline{t}+1/N}^{(N)} - M_{\underline{t}}^{(N)}| \xrightarrow{P} 0 \quad \text{as } N \rightarrow \infty \quad \text{for all } K > 0, \quad (3.2)$$

then $M^{(N)}$ is C -relatively compact in $D([0, \infty), \mathbb{R})$.

If, in addition,

$$\left\{ \left(M_{\underline{t}}^{(N)} \right)^2 + \langle M^{(N)} \rangle_{\underline{t}} : N \in \mathbb{N} \right\} \quad \text{is uniformly integrable for all } \underline{t},$$

then $M^{(N_k)} \xrightarrow{w} M$ in $D([0, \infty), \mathbb{R})$ implies M is a continuous L^2 -martingale and $(M^{(N_k)}, \langle M^{(N_k)} \rangle) \xrightarrow{w} (M, \langle M \rangle)$ in $D([0, \infty), \mathbb{R})$.

Lemma 3.5. *For any $\phi \in C_b^2(\mathbb{R})$, the sequence $C_t^{(N)}(\phi) \equiv \int_0^t X_s^{(N)}(A^N \phi) ds$ is C -relatively compact in $D([0, \infty), \mathbb{R})$.*

When we can verify the conditions of Lemma 3.4 for $M^{(b,N)}(\phi)$, and $M^{(e,N)}(\phi)$, the sequence $\{X^{(N)}(\phi) : N \in \mathbb{N}\}$ is C -relatively compact in $D([0, \infty), \mathbb{R})$. Moreover, if we check the condition of (i) in Theorem 3.2, then the tightness of $\{X^{(N)} : N \in \mathbb{N}\}$ follows immediately.

Before starting the proof of the above lemmas, we prepare the following lemma. It tells us the mean of the measure $X_{\underline{t}}^{(N)}$ is the same as the distribution of the scaled simple random walk.

Lemma 3.6. *We define historical process by*

$$H_t^{(N)} = \frac{1}{N} \sum_{\mathbf{x} \sim \underline{t}} \delta_{Y_{\frac{(\cdot \wedge \underline{t})N}{N^{1/2}}}^{\mathbf{x}}} \in \mathcal{M}_F(D([0, \infty), \mathbb{R})),$$

where $Y_s^{\mathbf{x}} = Y_s^{\mathbf{y}}$ for $0 \leq s < |\mathbf{x} \wedge \mathbf{y}| + 1$, that is $Y_s^{\mathbf{x}}$ is the position of the $\lfloor sN \rfloor$ -generation's ancestor of \mathbf{x} .

If $\psi : D([0, \infty), \mathbb{R}) \rightarrow \mathbb{R}_{\geq 0}$ is Borel, then for any $t \geq 0$

$$E \left[H_t^{(N)}(\psi) \right] = E_Y \left[\psi \left(\frac{Y_{(\cdot \wedge \underline{t})N}}{N^{\frac{1}{2}}} \right) \right], \quad (3.3)$$

where Y_{\cdot} is the trajectory of simple random walk on \mathbb{Z} . In particular, for all $\phi \in \mathcal{B}_+(\mathbb{R})$,

$$E \left[X_{\underline{t}}^{(N)}(\phi) \right] = E_Y \left[\phi \left(\frac{Y_{\underline{t}N}}{N^{\frac{1}{2}}} \right) \right]. \quad (3.4)$$

Moreover, for all $x, K > 0$, we have that

$$P_Y \left(\sup_{t \leq K} X_t^{(N)}(1) \geq x \right) \leq x^{-1}. \quad (3.5)$$

To prove this lemma, we introduce the notation. For $x(\cdot), y(\cdot) \in D([0, \infty), \mathbb{R})$ such that $y(0) = 0$,

$$(x/s/y)(t) = \begin{cases} x(t) & \text{if } 0 \leq t < s, \\ x(s) + y(t-s) & \text{if } t \geq s. \end{cases}$$

Then, $(x/s/y)(\cdot) \in D([0, \infty), \mathbb{R})$.

Proof. (3.3) follows from the Markov property. Indeed, we have

$$\begin{aligned} E \left[H_{\underline{t}}^{(N)}(\psi) \right] &= E \left[\frac{1}{N} \sum_{\mathbf{y} \sim \underline{t}} \psi \left(\frac{Y_{(\cdot \wedge \underline{t})N}^{\mathbf{y}}}{N^{\frac{1}{2}}} \right) \right] \\ &= E \left[\frac{1}{N} \sum_{\mathbf{x} \sim \underline{t} - \frac{1}{N}} \psi \left(\frac{Y_{(\cdot \wedge \underline{t})N}^{\mathbf{x}}}{N^{\frac{1}{2}}} \right) E \left[V^{\mathbf{x}} | \mathcal{F}_{\underline{t}N-1}^{(N)} \right] \right] \\ &= E \left[E_{Z_1} \left[\frac{1}{N} \sum_{\mathbf{x} \sim \underline{t} - \frac{1}{N}} \psi \left(\frac{\left(Y_{(\cdot \wedge (\underline{t} - \frac{1}{N}))N}^{\mathbf{x}} / \underline{t}N / Z_1 \right) ((\cdot \wedge \underline{t})N)}{N^{\frac{1}{2}}} \right) \right] \right], \end{aligned}$$

where $Z_1(\cdot)$ is a random function independent of $Y_{\underline{t}N}^{\mathbf{x}}$ such that $Z_1(s) = 0$ for $0 \leq s < 1$, $P(Z_1(s) = 1 \text{ for } s \geq 1) = P(Z_1(s) = -1 \text{ for } s \geq 1) = \frac{1}{2}$. Iterating

this,

$$\begin{aligned}
& E \left[H_{\underline{t}}^{(N)}(\psi) \right] \\
&= E \left[E_{Z_1, Z_2} \left[\frac{1}{N} \sum_{\mathbf{x} \sim \underline{t} - \frac{2}{N}} \psi \left(\frac{\left(\left(Y_{(\cdot \wedge \underline{t} - \frac{2}{N})N}^{\mathbf{x}} / \underline{t}N - 2 / Z_2 \right) / \underline{t}N - 1 / Z_1 \right) ((\cdot \wedge \underline{t})N)}{N^{\frac{1}{2}}} \right) \right] \right] \\
&= E_Y \left[\psi \left(\frac{Y_{((\cdot \wedge \underline{t})N)}}{N^{\frac{1}{2}}} \right) \right],
\end{aligned}$$

where Z_2 is independent copy of Z_1 and $Y(\cdot)$ is the trajectory of simple random walk. Also, (3.5) follows from the fact that $X_{\underline{t}}^{(N)}(1)$ is an $\mathcal{F}_{\underline{t}N}^{(N)}$ -martingale and from the L^1 inequality for non-negative submartingales and from (3.4). \square

Proof of Lemma 3.5. We know $X_0^{(N)}(\phi) = \phi(0)$. Also, we have that for any $K > 0$

$$\begin{aligned}
\left| C_t^{(N)}(\phi) - C_s^{(N)}(\phi) \right| &\leq \int_{\underline{s}}^{\underline{t}} \left| X_{\underline{u}}^{(N)}(A^N \phi) \right| du \\
&\leq \sup_{\underline{u} \leq K} C(\phi) X_{\underline{u}}^{(N)}(1) |\underline{t} - \underline{s}|.
\end{aligned} \tag{3.6}$$

We can use the Arzela-Ascoli Theorem by (3.5) and (3.6) so that $\{C^{(N)}(\phi) : N \in \mathbb{N}\}$ are C -relatively compact sequences in $D([0, \infty), \mathbb{R})$. \square

Proof of Lemma 3.3. Let $h_N(y) = E^y \left[\left(\phi \left(\frac{Y_1}{N^{\frac{1}{2}}} \right) - \phi \left(\frac{Y_0}{N^{\frac{1}{2}}} \right) \right)^2 \right]$. First, we remark that

$$\phi \left(\frac{Y_{\underline{s}N+1}^{\mathbf{x}}}{N^{\frac{1}{2}}} \right) - \phi \left(\frac{Y_{\underline{s}N}^{\mathbf{x}}}{N^{\frac{1}{2}}} \right) - \frac{1}{N} A^N \phi \left(\frac{Y_{\underline{s}N}^{\mathbf{x}}}{N^{\frac{1}{2}}} \right)$$

are orthogonal for $\mathbf{x} \neq \mathbf{x}' \sim \underline{s}$. Since $M_t^{(s,N)}(\phi)$ is a martingale, we have that

$$\begin{aligned}
& E \left[\left(M_K^{(s,N)}(\phi) \right)^2 \right] \\
&= \frac{1}{N^2} \sum_{\underline{s} < K} E \left[\left(\Delta M_{\underline{s}}^{(s,N)}(\phi) \right)^2 \right] \\
&= \frac{1}{N^2} \sum_{\underline{s} < K} E \left[\sum_{\mathbf{x} \sim \underline{s}} E \left[\left(\phi \left(\frac{Y_{\underline{s}N+1}^{\mathbf{x}}}{N^{\frac{1}{2}}} \right) - \phi \left(\frac{Y_{\underline{s}N}^{\mathbf{x}}}{N^{\frac{1}{2}}} \right) - \frac{1}{N} A^N \phi \left(\frac{Y_{\underline{s}N}^{\mathbf{x}}}{N^{\frac{1}{2}}} \right) \right)^2 \middle| \mathcal{F}_{\underline{s}N}^{(N)} \right] \right] \\
&\leq \frac{2}{N} \sum_{\underline{s} < K} E \left[\frac{1}{N} \sum_{\mathbf{x} \sim \underline{s}} \left(h_N \left(\frac{Y_{\underline{s}N}^{\mathbf{x}}}{N^{\frac{1}{2}}} \right) + \frac{1}{N^2} \|A^N \phi\|^2 \right) \right] \\
&\leq 2E \left[\int_0^K \left(X_s^{(N)}(h_N) + \|A^N \phi\|_\infty^2 N^{-2} X_s^{(N)}(1) \right) ds \right] \\
&\leq 2 \left(E_Y \left[\int_0^K \left(\phi \left(\frac{Y_{\underline{s}N+1}}{N^{\frac{1}{2}}} \right) - \phi \left(\frac{Y_{\underline{s}N}}{N^{\frac{1}{2}}} \right) \right)^2 ds \right] + \frac{K}{N^2} \sup_N \|A^N \phi\|_\infty^2 X_0^{(N)}(1) \right) \\
&\rightarrow 0,
\end{aligned}$$

where we have used Lemma 3.6 and the fact that $\sup_N \|A^N \phi\|_\infty < \infty$ for $\phi \in C_b^2(\mathbb{R})$ and $\{X_{\underline{t}}^{(N)}(1) : 0 \leq \underline{t} \leq \underline{K}\}$ is martingale with respect to $\mathcal{F}_{\underline{t}N}^{(N)}$ in the last line. \square

Next, we will check the conditions in Lemma 3.4 for $M^{(b,N)}(\phi)$ and $M^{(e,N)}(\phi)$, that is,

- (1) $\left\{ \left\langle M^{(b,N)}(\phi) \right\rangle_{\cdot} + \left\langle M^{(e,N)}(\phi) \right\rangle_{\cdot} : N \in \mathbb{N} \right\}$ is C -relatively compact in $D([0, \infty), \mathbb{R})$.
- (2) $\sup_{0 \leq \underline{t} \leq K} \left| M_{\underline{t} + \frac{1}{N}}^{(b,N)}(\phi) - M_{\underline{t}}^{(b,N)}(\phi) + M_{\underline{t} + \frac{1}{N}}^{(e,N)}(\phi) - M_{\underline{t}}^{(e,N)}(\phi) \right| \xrightarrow{P} 0$ as $N \rightarrow \infty$ for all $K > 0$.
- (3) $\left\{ \left(M_{\underline{t}}^{(b,N)}(\phi) \right)^2 + \left(M_{\underline{t}}^{(e,N)}(\phi) \right)^2 + \left\langle M^{(b,N)}(\phi) \right\rangle_{\underline{t}} + \left\langle M^{(e,N)}(\phi) \right\rangle_{\underline{t}} : N \in \mathbb{N} \right\}$ is uniformly integrable for all \underline{t} .

As we verified that $M^{(b,N)}(\phi)$ and $M^{(e,N)}(\phi)$ are orthogonal, we have that

$$\left\langle M^{(b,N)}(\phi) + M^{(e,N)}(\phi) \right\rangle_{\cdot} = \left\langle M^{(b,N)}(\phi) \right\rangle_{\cdot} + \left\langle M^{(e,N)}(\phi) \right\rangle_{\cdot}.$$

Moreover, since under fixed environment $\{\xi(n, x) : (n, x) \in \mathbb{N} \times \mathbb{Z}\}$, $V^{\mathbf{x}}$ and

V^y are independent for $x \neq y$, we have that

$$\begin{aligned}
& \left\langle M^{(b,N)}(\phi) \right\rangle_{\underline{t}} \\
&= \sum_{\underline{s} < t} E \left[\left(M_{\underline{s} + \frac{1}{N}}^{(b,N)}(\phi) - M_{\underline{s}}^{(b,N)}(\phi) \right)^2 \middle| \mathcal{F}_{\underline{s}N}^{(N)} \right] \\
&= \frac{1}{N^2} \sum_{\underline{s} < t} \sum_{x \sim \underline{t}} E \left[\phi \left(\frac{Y_{\underline{t}N+1}^x}{N^{\frac{1}{2}}} \right)^2 \middle| \mathcal{F}_{\underline{t}N}^{(N)} \right] E \left[\left(V^x - 1 - \frac{\beta \xi(\underline{t}N, Y_{\underline{t}N}^x)}{N^{\frac{1}{4}}} \right)^2 \middle| \mathcal{F}_{\underline{t}N}^{(N)} \right] \\
&= \frac{1}{N} \sum_{\underline{s} < t} X_{\underline{t}}^{(N)}(\phi^2) \left(1 - \frac{\beta^2}{N^{\frac{1}{2}}} \right) \left(1 + \mathcal{O}(N^{-\frac{1}{2}}) \right) \\
&= \left(1 + \mathcal{O}(N^{-\frac{1}{2}}) \right) \int_0^t X_s^{(N)}(\phi^2) ds,
\end{aligned}$$

and

$$\begin{aligned}
& \left\langle M^{(e,N)}(\phi) \right\rangle_{\underline{t}} \\
&= \sum_{\underline{s} < t} E \left[\left(M_{\underline{s} + \frac{1}{N}}^{(e,N)}(\phi) - M_{\underline{s}}^{(e,N)}(\phi) \right)^2 \middle| \mathcal{F}_{\underline{s}N}^{(N)} \right] \\
&= \frac{\beta^2}{N^2} \sum_{\underline{s} < t} \sum_{x, \tilde{x} \sim \underline{t}} E \left[\phi \left(\frac{Y_{\underline{t}N+1}^x}{N^{\frac{1}{2}}} \right) \phi \left(\frac{Y_{\underline{t}N+1}^{\tilde{x}}}{N^{\frac{1}{2}}} \right) \middle| \mathcal{F}_{\underline{t}N}^{(N)} \right] \frac{\mathbf{1}_{\{Y_{\underline{t}N}^x = Y_{\underline{t}N}^{\tilde{x}}\}}}{N^{\frac{1}{2}}} \\
&= \frac{\beta^2}{N^2} \sum_{\underline{s} < t} \sum_{x, \tilde{x} \sim \underline{t}} \phi \left(\frac{Y_{\underline{t}N}^x}{N^{\frac{1}{2}}} \right)^2 \frac{\mathbf{1}_{\{Y_{\underline{t}N}^x = Y_{\underline{t}N}^{\tilde{x}}\}}}{N^{\frac{1}{2}}} \left(1 + \mathcal{O}(N^{-\frac{1}{2}}) \right) \\
&= \frac{1 + \mathcal{O}(N^{-\frac{1}{2}})}{N} \beta^2 \sum_{\underline{s} < t} \sum_{x \in \mathbb{Z}} \phi \left(\frac{x}{N^{\frac{1}{2}}} \right)^2 \frac{(B_{\underline{s}N, x}^{(N)})^2}{N^{\frac{3}{2}}} \\
&= \left(1 + \mathcal{O}(N^{-\frac{1}{2}}) \right) \beta^2 \int_0^t \sum_{x \in \mathbb{Z}} \phi \left(\frac{x}{N^{\frac{1}{2}}} \right)^2 \frac{(B_{\underline{s}N, x}^{(N)})^2}{N^{\frac{3}{2}}} ds,
\end{aligned}$$

where $|\mathcal{O}(N^{-\frac{1}{2}})| \leq C_\phi N^{-\frac{1}{2}}$ for a constant C_ϕ that depends only on ϕ .

Therefore, we have that

$$\begin{aligned}
& \left\langle M^{(b,N)}(\phi) \right\rangle_{\underline{t}} + \left\langle M^{(e,N)}(\phi) \right\rangle_{\underline{t}} - \left\langle M^{(b,N)}(\phi) \right\rangle_{\underline{s}} - \left\langle M^{(e,N)}(\phi) \right\rangle_{\underline{s}} \\
&\leq C_\phi \left(\left\langle M^{(b,N)}(1) \right\rangle_{\underline{t}} + \left\langle M^{(e,N)}(1) \right\rangle_{\underline{t}} - \left\langle M^{(b,N)}(1) \right\rangle_{\underline{s}} - \left\langle M^{(e,N)}(1) \right\rangle_{\underline{s}} \right) \\
&= C \left(\left\langle X^{(N)}(1) \right\rangle_{\underline{t}} - \left\langle X^{(N)}(1) \right\rangle_{\underline{s}} \right), \tag{3.7}
\end{aligned}$$

where we remark that $\{X_{\underline{t}}^{(N)}(1) : 0 \leq \underline{t}\}$ is a martingale with respect to $\mathcal{F}_{\underline{t}N}^{(N)}$.

We will prove C -relative compactness of (3.7) by showing the following lemma.

Lemma 3.7. *For any $K > 0$*

$$\sup_N E \left[\left(X_{\underline{K}}^{(N)}(1) \right)^2 \right] < \infty.,$$

and for any $\varepsilon > 0$,

$$\lim_{\delta \rightarrow 0} \sup_{N \geq 1} P \left(\sup_{0 \leq s \leq K} \left(\left\langle X^{(N)}(1) \right\rangle_{\underline{s}+\delta} - \left\langle X^{(N)}(1) \right\rangle_{\underline{s}} \right) > \varepsilon \right) = 0.$$

Proof. We remark that for each N , $B_n^{(N)}$ is a martingale with respect to the filtration $\mathcal{F}_n^{(N)}$.

Let $B_n^{(i,N)}$ be the total number of particles at time n which are the descendants from i -th initial particle. Then, we remark that for $i \neq j$

$$\begin{aligned} E \left[B_{\lfloor KN \rfloor}^{(i,N)} B_{\lfloor KN \rfloor}^{(j,N)} \right] &= E \left[E \left[B_{\lfloor KN \rfloor}^{(i,N)} \middle| \mathcal{H} \right] E \left[B_{\lfloor KN \rfloor}^{(j,N)} \middle| \mathcal{H} \right] \right] \\ &= E_{Y^1 Y^2} \left[\left(1 + \frac{\beta^2}{N^{\frac{1}{2}}} \right)^{\#\{i \leq \lfloor KN \rfloor : Y_i^1 = Y_i^2\}} \right], \end{aligned}$$

where \mathcal{H} is the σ -algebra generated by $\{\xi(n, x) : (n, x) \in \mathbb{N} \times \mathbb{Z}\}$, and Y^1 and Y^2 are independent simple random walks on \mathbb{Z} starting from the origin.

On the other hand,

$$\begin{aligned} E \left[\left(B_{\lfloor KN \rfloor}^{(i,N)} \right)^2 \right] &= 1 + \sum_{k=1}^{\lfloor KN \rfloor - 1} c E_{Y^1 Y^2} \left[\left(1 + \frac{\beta^2}{N^{\frac{1}{2}}} \right)^{\#\{k < i \leq \lfloor KN \rfloor : Y_i^1 = Y_i^2\}} : Y_k^1 = Y_k^2 \right] + c \\ &\leq \lfloor KN \rfloor E_{Y^1 Y^2} \left[\left(1 + \frac{\beta^2}{N^{\frac{1}{2}}} \right)^{\#\{i \leq \lfloor KN \rfloor : Y_i^1 = Y_i^2\}} \right], \end{aligned}$$

where $c = 1 - \frac{1}{N^{\frac{1}{2}}} < 1$ [28, Lemma 2.3]. Thus, we have that

$$\begin{aligned} E \left[\left(X_{\underline{K}}^{(N)}(1) \right)^2 \right] &\leq \frac{1}{N^2} (N(N-1) + N \lfloor KN \rfloor) E_{Y^1 Y^2} \left[\left(1 + \frac{\beta^2}{N^{\frac{1}{2}}} \right)^{\#\{i \leq \lfloor KN \rfloor : Y_i^1 = Y_i^2\}} \right] \\ &\leq C(K) E_{Y^1 Y^2} \left[\left(1 + \frac{\beta^2}{N^{\frac{1}{2}}} \right)^{\#\{i \leq \lfloor KN \rfloor : Y_i^1 = Y_i^2\}} \right]. \end{aligned}$$

Since $E_{Y^1 Y^2} \left[\left(1 + \frac{\beta^2}{N^{\frac{1}{2}}} \right)^{\#\{i \leq \lfloor KN \rfloor : Y_i^1 = Y_i^2\}} \right]$ is bounded (Lemma 5.1), we complete the proof.

Now, we turn to the proof of the latter part of the statement. Let $\delta > 0$. It follows from the above argument that

$$\begin{aligned} &\left\langle X^{(N)}(1) \right\rangle_t - \left\langle X^{(N)}(1) \right\rangle_s \\ &= \int_s^t \left(X_u^{(N)}(1) + \beta^2 \sum_{x \in \mathbb{Z}} \frac{\left(B_{\lfloor uN \rfloor, x}^{(N)} \right)^2}{N^{\frac{3}{2}}} \right) du. \end{aligned}$$

We know that $\left| \int_s^t X_u^{(N)}(1) du \right| \leq \left(\sup_{u \leq K} X_u^{(N)}(1) \right) |t - s|$ and Lemma 3.6 implies that this term converges in probability to 0 as $|t - s| \rightarrow 0$ uniformly in $0 \leq s \leq t \leq K$. So, it is enough to show that for any $\varepsilon > 0$

$$\lim_{\delta \rightarrow 0} \sup_{N \geq 1} P \left(\sup_{0 \leq s \leq K} \int_s^{s+\delta} \sum_{x \in \mathbb{Z}} \frac{\left(B_{[uN],x}^{(N)} \right)^2}{N^{\frac{3}{2}}} du > \varepsilon \right) = 0.$$

We consider the segments $I_k^\delta = [2k\delta, 2(k+1)\delta]$ for $0 \leq k \leq \lfloor \frac{K}{2\delta} \rfloor$. Then, we have by Corollary 5.3 that

$$\begin{aligned} & E \left[\left(\int_{I_k^\delta} \sum_{x \in \mathbb{Z}} \frac{\left(B_{[uN],x}^{(N)} \right)^2}{N^{\frac{3}{2}}} du \right)^2 \right] \\ &= \frac{1}{N^5} E \left[\sum_{s=2k\delta N}^{2(k+1)\delta N} \sum_{t=2k\delta N}^{2(k+1)\delta N} \sum_{x,y \in \mathbb{Z}} \left(B_{[sN],x}^{(N)} \right)^2 \left(B_{[tN],y}^{(N)} \right)^2 \right] \\ &\leq \frac{1}{N^5} \left(\sum_{s=2k\delta N}^{2(k+1)\delta N} \sum_{x \in \mathbb{Z}} E \left[\left(B_{[sN],x}^{(N)} \right)^4 \right]^{\frac{1}{2}} \right)^2. \end{aligned} \quad (3.8)$$

Corollary 5.3 implies that

$$E \left[\left(B_{[sN],x}^{(N)} \right)^4 \right] \leq (s \vee 1)^4 N^4 E_{Y^1 Y^2 Y^3 Y^4} \left[\left(1 + \frac{7\beta^2}{N^{\frac{1}{2}}} \right)^{\#\{1 \leq i \leq sN : Y_i^a = Y_i^b, a,b \in \{1,2,3,4\}\}} \cdot \begin{matrix} Y_{[sN]}^a = x, \\ a \in \{1,2,3,4\} \end{matrix} \right], \quad (3.9)$$

where we have used that for N large enough, $E \left[\left(1 + \frac{\beta \xi(0,0)}{N^{\frac{1}{4}}} \right)^4 \right] \leq 1 + \frac{7\beta^2}{N^{\frac{1}{2}}}$.

Hölder's inequality and Lemma 5.1 imply that

$$\begin{aligned} (3.9) &\leq (s \vee 1)^4 N^4 E_{Y^1 Y^2} \left[\left(1 + \frac{7\beta^2}{N^{\frac{1}{2}}} \right)^{6\#\{1 \leq i \leq sN : Y_i^1 = Y_i^2\}} : Y_{[sN]}^1 = Y_{[sN]}^2 = x \right] P_{Y^1} \left(Y_{[sN]}^1 = x \right)^2 \\ &\leq \frac{(s \vee 1)^4 N^4}{(sN \vee 1)^{\frac{1}{2}}} P_{Y^1} \left(Y_{[sN]}^1 = x \right)^3. \end{aligned}$$

Thus, local limit theorem implies that

$$\begin{aligned} (3.8) &\leq \frac{C}{N} \left(\sum_{s=2k\delta N}^{2(k+1)\delta N} \sum_{x \in \mathbb{Z}} \frac{(K \vee 1)^2}{(sN \vee 1)^{\frac{1}{4}}} \frac{1}{(sN \vee 1)^{\frac{1}{4}}} P_{Y^1} \left(Y_{[sN]}^1 = x \right) \right)^2 \\ &\leq \frac{CK^4}{N} \left(\sqrt{2(k+1)\delta N} - \sqrt{2k\delta N} \right)^2. \end{aligned}$$

Thus, we obtained that

$$P \left(\int_{I_k^\delta} \sum_{x \in \mathbb{Z}} \frac{\left(B_{[uN],x}^{(N)} \right)^2}{N^{\frac{3}{2}}} du > \varepsilon \right) \leq \frac{CK^4 \delta}{\varepsilon^2 (\sqrt{2(k+1)} + \sqrt{2k})^2}.$$

Since for each $0 \leq s \leq K$, there is some k such that $[s, s + \delta] \subset I_k^\delta \cup I_{k+1}^\delta$, we have that

$$\begin{aligned} \sup_{N \geq 1} P \left(\sup_{0 \leq s \leq K} \int_s^{s+\delta} \sum_{x \in \mathbb{Z}} \frac{(B_{[uN],x}^{(N)})^2}{N^{\frac{3}{2}}} du > \varepsilon \right) &\leq 2 \sum_{k=0}^{\frac{K}{\delta}} \frac{CK^4 \delta}{\varepsilon^2 (\sqrt{2(k+1)} + \sqrt{2k})^2} \\ &\leq 2 \frac{CK^4 \delta \log \frac{K}{\delta}}{\varepsilon^2} \rightarrow 0 \text{ as } \delta \rightarrow 0. \end{aligned}$$

□

Also, we prove the following lemmas to check the conditions (1)-(3).

Lemma 3.8. For $\phi \in C_b^2(\mathbb{R})$,

$$\lim_{N \rightarrow \infty} E \left[\sum_{t \leq K} |\Delta M_t^{(b,N)}(\phi) + \Delta M_t^{(e,N)}(\phi)|^4 \right] = 0 \quad \text{for all } K > 0.$$

Lemma 3.9. For $\phi \in C_b^2(\mathbb{R})$,

$$\sup_N E \left[\sup_{t \leq K} \left| M_t^{(b,N)}(\phi) + M_t^{(e,N)}(\phi) \right|^4 \right] < \infty \quad \text{for all } K > 0,$$

and

$$E \left[\left(\left\langle M^{(b,N)}(\phi) + M^{(e,N)}(\phi) \right\rangle_K \right)^2 \right] < \infty \quad \text{for all } K > 0.$$

If we prove these lemmas, then we can verify the condition of Theorem 3.2 (ii).

Proof of the C -relatively compactness of $\{X^{(N)}(\phi) : N \in \mathbb{N}\}$. When we look at the process $\{X^{(N)}(\phi)\}$, it is divided into some processes, $X_0^{(N)}(\phi)$, $M^{(b,N)}(\phi)$, $M^{(e,N)}(\phi)$, $M^{(b,N)}(\phi)$, and $C^{(N)}(\phi)$.

We know that $M^{(b,N)}(\phi)$ and $X_0^{(N)}(\phi)$ converges to constant by Assumptions and Lemma 3.3. C -relative compactness of $C^{(N)}(\phi)$ has been proved in Lemma 3.5.

Arzela-Ascoli's theorem and Lemma 3.7 imply that $\left\{ \left\langle M^{(b,N)}(\phi) + M^{(e,N)}(\phi) \right\rangle : N \in \mathbb{N} \right\}$ is C -relatively compact in $D([0, \infty), \mathbb{R})$. Also, (3.2) follows from Lemma 3.8.

The uniform integrability of $\left\{ \left(M_t^{(b,N)}(\phi) + M_t^{(e,N)}(\phi) \right)^2 + \left\langle M^{(b,N)}(\phi) + M^{(e,N)}(\phi) \right\rangle_t \right\}$ has been shown by Lemma 3.7 and Lemma 3.9. Thus, we have checked all conditions in Lemma 3.4 so that $\left\{ M^{(b,N)}(\phi) + M^{(e,N)}(\phi), \left\langle M^{(b,N)}(\phi) + M^{(e,N)}(\phi) \right\rangle \right\}$ is C -relatively compact in $D([0, \infty), \mathbb{R})$.

Thus, $\{X^{(N)}(\phi)\}$ is C -relatively compact in $D([0, \infty), \mathbb{R})$ for each $\phi \in C_b^2(\mathbb{R})$.

□

To prove Lemma 3.8, we will use the following proposition (see [3]).

Proposition 3.10. *Let $\phi : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is continuous, increasing, $\phi(0) = 0$ and $\phi(2\lambda) \leq c_0\phi(\lambda)$ for all $\lambda \geq 0$. (M_n, \mathcal{F}_n) is a martingale, $M_n^* = \sup_{k \leq n} |M_k|$, $\langle M \rangle_n = \sum_{i=1}^n E[(M_i - M_{i-1})^2 | \mathcal{F}_{i-1}] + E[M_0^2]$, and $d_n^* = \max_{1 \leq k \leq n} |M_k - M_{k-1}|$. Then, there exists $c = c(c_0)$ such that*

$$E[\phi(M_n^*)] \leq cE\left[\phi\left(\langle M \rangle_n^{1/2}\right) + \phi(d_n^*)\right].$$

Proof of Lemma 3.8. It is enough to show that

$$\lim_{N \rightarrow \infty} E\left[\sum_{t \leq K} \left|\Delta M_t^{(b,N)}(\phi)\right|^4 + \left|\Delta M_t^{(e,N)}(\phi)\right|^4\right] = 0 \quad \text{for all } K > 0.$$

Conditional on $\mathcal{G}_{tN}^{(N)}$, $\Delta M_t^{(b,N)}(\phi)$ is a sum of mean 0 independent random variables; $W^{(b,x,N)} := \frac{1}{N}\phi\left(\frac{Y_{tN+1}^x}{N^{\frac{1}{2}}}\right) \left(V^x - 1 - \frac{\beta\xi\left(tN, Y_{tN}^x\right)}{N^{\frac{1}{4}}}\right)$. Applying Proposition 3.10 into $\sum_{x \sim t} W^{(b,x,N)}$, we have

$$E\left[\left(\sup_{i \leq B_{tN}^{(N)}} \sum_{k=1}^i W^{(b,x_k,N)}\right)^4 \middle| \mathcal{G}_{tN}^{(N)}\right] \leq c \left(\sum_{i \leq B_{tN}^{(N)}} \left(\frac{C_1(\phi)(1 - \mathcal{O}(N^{-1/2}))}{N^2}\right)^2 + \left(\frac{C_2(\phi)}{N}\right)^4\right).$$

Thus,

$$E\left[\sum_{t \leq K} \left|\Delta M_t^{(b,N)}(\phi)\right|^4\right] \leq c \left(\frac{C_1(\phi)^2(1 - \mathcal{O}(N^{-1/2}))}{N^4} \cdot (KN) \cdot E[NX_t^{(N)}(1)] + KN \cdot \frac{C_2(\phi)^4}{N^4}\right) \rightarrow 0.$$

Next, we will prove that

$$\lim_{N \rightarrow \infty} E\left[\sum_{t \leq K} \left|\Delta M_t^{(e,N)}(\phi)\right|^4\right] = 0 \quad \text{for all } K > 0.$$

It is clear that for $\phi \in C_b^2(\mathbb{R})$

$$E\left[\left|\Delta M_t^{(e,N)}(\phi)\right|^4\right] \leq C(\phi)E\left[\sum_{x,y \in \mathbb{Z}} 2 \frac{\left(B_{tN,x}^{(N)}\right)^2 \left(B_{tN,y}^{(N)}\right)^2}{N^5}\right].$$

Then, it follows from Corollary 5.3 and the similar argument in the proof of

Lemma 3.7 that

$$\begin{aligned}
& \frac{E \left[\left(B_{\underline{t}N,x}^{(N)} \right)^2 \left(B_{\underline{t}N,y}^{(N)} \right)^2 \right]}{N^5} \\
& \leq \frac{C(\underline{t} \vee 1)^4}{N} E_{Y^1 Y^2 Y^3 Y^4} \left[\left(1 + \frac{7\beta^2}{N^{\frac{1}{2}}} \right)^{\#\{1 \leq i \leq \underline{t}N : Y_i^a = Y_i^b, a,b \in \{1,2,3,4\}\}} : \begin{matrix} Y_{\underline{t}N}^1 = Y_{\underline{t}N}^2 = x \\ Y_{\underline{t}N}^3 = Y_{\underline{t}N}^4 = y \end{matrix} \right] \\
& \leq \frac{C(\underline{t} \vee 1)^4}{N} \prod_{\substack{a,b \in \{1,2,3,4\} \\ a \neq b}} E_{Y^1 Y^2 Y^3 Y^4} \left[\left(1 + \frac{7\beta^2}{N^{\frac{1}{2}}} \right)^{6\#\{1 \leq i \leq \underline{t}N : Y_i^a = Y_i^b\}} : \begin{matrix} Y_{\underline{t}N}^1 = Y_{\underline{t}N}^2 = x \\ Y_{\underline{t}N}^3 = Y_{\underline{t}N}^4 = y \end{matrix} \right]^{\frac{1}{6}} \\
& \leq \frac{C(\underline{t} \vee 1)^4}{N\sqrt{\underline{t}N}} P_{Y^1} \left(Y_{\underline{t}N}^1 = x \right) P_{Y^1} \left(Y_{\underline{t}N}^1 = y \right) \left(P_{Y^1} \left(Y_{\underline{t}N}^1 = x \right) \wedge P_{Y^1} \left(Y_{\underline{t}N}^1 = y \right) \right).
\end{aligned}$$

Thus, we have that

$$E \left[\left| \Delta M_{\underline{t}}^{(e,N)}(\phi) \right|^4 \right] \leq C(\phi)(K \vee 1)^4 \sum_{\underline{t} \leq K} \frac{1}{N \cdot \underline{t}N} \rightarrow 0,$$

as $N \rightarrow \infty$. □

Proof of Lemma 3.9. We apply Proposition 3.10 into martingale $M_{\underline{t}}^{(b,N)}(\phi) + M_{\underline{t}}^{(e,N)}(\phi)$. Then, we have that

$$\begin{aligned}
E \left[\sup_{\underline{t} \leq K} \left(M_{\underline{t}}^{(b,N)}(\phi) + M_{\underline{t}}^{(e,N)}(\phi) \right)^4 \right] & \leq c(\phi) \left(E \left[\left(\left\langle M^{(b,N)}(1) \right\rangle_K + \left\langle M^{(e,N)}(1) \right\rangle_K \right)^2 \right] \right. \\
& \quad \left. + \sum_{\underline{t} \leq K} \left(\left| \Delta M_{\underline{t}}^{(b,N)}(1) \right|^4 + \left| \Delta M_{\underline{t}}^{(e,N)}(1) \right|^4 \right) \right).
\end{aligned}$$

The second term in the right hand side goes to 0 as $N \rightarrow \infty$ by Lemma 3.8.

The first term is bounded above by

$$CE \left[\sum_{\underline{s}, \underline{t} \leq K} \left(\frac{X_{\underline{s}}^{(N)}(1) X_{\underline{t}}^{(N)}(1)}{N^2} + \beta^4 \sum_{x,y \in \mathbb{Z}} \frac{\left(B_{\underline{t}N,x}^{(N)} \right)^2 \left(B_{\underline{s}N,y}^{(N)} \right)^2}{N^{\frac{3}{2}} N^{\frac{3}{2}}} \right) \right].$$

Since $X_{\underline{t}}^{(N)}(1)$ is a martingale, $E \left[X_{\underline{s}}^{(N)}(1) X_{\underline{t}}^{(N)}(1) \right] = E \left[X_{\underline{s}}^{(N)}(1) X_{\underline{s}}^{(N)}(1) \right]$ for $\underline{s} \leq \underline{t}$. Thus,

$$E \left[\sum_{\underline{s}, \underline{t} \leq K} \frac{X_{\underline{s}}^{(N)}(1) X_{\underline{t}}^{(N)}(1)}{N^2} \right] \leq K^2 E \left[\left(X_{\underline{K}}^{(N)}(1) \right)^2 \right]$$

is bounded in N for all K by Lemma 3.7.

Also, we know that from the proof of Lemma 3.7 that

$$\sum_{\underline{s}, \underline{t} \leq K} E \left[\sum_{x,y \in \mathbb{Z}} \frac{\left(B_{\underline{s}N,x}^{(N)} \right)^2 \left(B_{\underline{t}N,y}^{(N)} \right)^2}{N^5} \right] \leq \frac{CK^4}{N} \left(\sqrt{KN} \right)^2 < \infty.$$

□

In the end of this subsection, we complete the proof of the tightness by checking the condition (i) in Theorem 3.2. The proof follows the one in [23, p155]

Check for (i) in Theorem 3.2. Let $\varepsilon, T > 0$ and $\eta(\varepsilon) > 0$ (η will be chosen later). Let $K_0 \subset D([0, \infty), \mathbb{R})$ be a compact set such that $\sup_N P\left(\frac{Y_{\cdot N}}{N^{\frac{1}{2}}} \in K_0^c\right) < \eta$. Let $K_T = \{y_t, y_{t-} : t \leq T, y \in K_0\}$. Then, K_T is compact in \mathbb{R} . Clearly,

$$\sup_N P\left(\frac{Y_{Nt}}{N^{\frac{1}{2}}} \in K_T^c \text{ for some } t \leq T\right) < \eta.$$

Let

$$\begin{aligned} R_t^{(N)} &= H_t^{(N)}(y : y(s) \in K_T^c \text{ for some } s \leq t) \\ &= \frac{1}{N} \sum_{\mathbf{x} \sim \underline{t}} \sup_{\underline{s} \leq \underline{t}} \mathbf{1}_{K_T^c} \left(\frac{Y_{\underline{s}N}^{\mathbf{x}}}{N^{\frac{1}{2}}} \right). \end{aligned}$$

First, we will claim that $R_t^{(N)}$ is an $\mathcal{F}_{\underline{t}N}^{(N)}$ -submartingale. Clearly, $R_t^{(N)}$ is constant on $[\underline{t}, \underline{t} + \frac{1}{N})$. So, it is enough to show that

$$E \left[R_{\underline{t} + \frac{1}{N}}^{(N)} - R_{\underline{t}}^{(N)} \middle| \mathcal{F}_{\underline{t}N}^{(N)} \right] \geq 0 \quad \text{a.s.} \quad (3.10)$$

We have

$$\begin{aligned} R_{\underline{t} + \frac{1}{N}}^{(N)} - R_{\underline{t}}^{(N)} &= \frac{1}{N} \sum_{\mathbf{x} \sim \underline{t}} \sup_{\underline{s} \leq \underline{t} + \frac{1}{N}} \mathbf{1}_{K_T^c} \left(\frac{Y_{\underline{s}N}^{\mathbf{x}}}{N^{\frac{1}{2}}} \right) V^{\mathbf{x}} - \sup_{\underline{s} \leq \underline{t}} \mathbf{1}_{K_T^c} \left(\frac{Y_{\underline{s}N}^{\mathbf{x}}}{N^{\frac{1}{2}}} \right) \\ &\geq \frac{1}{N} \sum_{\mathbf{x} \sim \underline{t}} (V^{\mathbf{x}} - 1) \sup_{\underline{s} \leq \underline{t}} \mathbf{1}_{K_T^c} \left(\frac{Y_{\underline{s}N}^{\mathbf{x}}}{N^{\frac{1}{2}}} \right). \end{aligned}$$

The conditional expectation of the last term with respect to $\mathcal{F}_{\underline{t}N}^{(N)}$ is equal to 0. Thus, (3.10) is proved. Now we apply L^1 -inequality for submartingale into $R_t^{(N)}$ so that

$$\begin{aligned} P \left(\sup_{\underline{s} \leq T} X_{\underline{s}}^{(N)}(K_T^c) > \varepsilon \right) &\leq P \left(\sup_{t \leq T} R_t^{(N)} > \varepsilon \right) \\ &\leq \varepsilon^{-1} E[R_{\underline{T}}^{(N)}] \\ &\leq \varepsilon^{-1} P \left(\frac{Y_{sN}}{N^{\frac{1}{2}}} \in K_T^c, \text{ for some } s \leq T \right) \leq \varepsilon \end{aligned}$$

by taking $\eta(\varepsilon) = \varepsilon^2$. □

3.2 Identification of the limit point process

From the lemmas in section 3.1, we know that for $\phi \in C_b^2(\mathbb{R})$, each term of

$$Z_{\underline{t}}^{(N)}(\phi) = X_{\underline{t}}^{(N)}(\phi) - \phi(0) - \int_0^{\underline{t}} X_{\underline{s}}^{(N)}(A^N \phi) ds, \quad (3.11)$$

and

$$\left\langle Z^{(N)}(\phi) \right\rangle_{\underline{t}} = \left\langle M^{(b,N)}(\phi) \right\rangle_{\underline{t}} + \left\langle M^{(e,N)}(\phi) \right\rangle_{\underline{t}} + \left\langle M^{(s,N)}(\phi) \right\rangle_{\underline{t}}$$

are C -relatively compact in $D([0, \infty), \mathbb{R})$ and we found by from Lemma 3.4 that the limit points satisfy

$$Z_t(\phi) = X_t(\phi) - \phi(0) - \int_0^t \frac{1}{2} X_s(\Delta\phi) ds$$

and

$$\langle Z(\phi) \rangle_t = \int_0^t X_s(\phi^2) ds + M_t^{(e)}(\phi),$$

where $M_t^{(e)}(\phi)$ is a limit point of $M_{\underline{t}}^{(e,N)}(\phi)$. Therefore, we need to identify $M_t^{(e)}(\phi)$.

First, we give an approximation of $X_t^{(N)}$ by some measure valued processes which have densities. For $(t, y) \in \mathbb{R}_{\geq 0} \times \mathbb{R}$, we define $u^{(N)}(t, y)$ by

$$u^{(N)}(t, y) = \frac{B_{tN,x}^{(N)}}{2\sqrt{N}} \quad \text{for } \underline{t} \leq t < \underline{t} + \frac{1}{N} \text{ and } y \in \left[\frac{x-1}{N^{\frac{1}{2}}}, \frac{x+1}{N^{\frac{1}{2}}} \right), \quad x \in \mathbb{Z}.$$

Actually, integrating $u^{(N)}(t, y)$ over $\left[\frac{x-1}{N^{\frac{1}{2}}}, \frac{x+1}{N^{\frac{1}{2}}} \right)$ for each $x \in \mathbb{Z}$, they coincide with $\frac{B_{tN,x}^{(N)}}{N}$. Thus, we can regard $u^{(N)}(t, y)$ as an approximation of $X^{(N)}$.

Also, $\langle M^{(e,N)}(\phi) \rangle_{\underline{t}}$ can be rewritten as

$$\begin{aligned} \left\langle M^{(e,N)}(\phi) \right\rangle_{\underline{t}} &= \int_0^{\underline{t}} \sum_{x \in \mathbb{Z}} \phi \left(\frac{x}{N^{\frac{1}{2}}} \right)^2 \frac{\beta^2 \left(B_{\lfloor sN \rfloor, x}^{(N)} \right)^2}{N^{\frac{3}{2}}} ds \\ &= 2\beta^2(1 + \mathcal{O}(N^{-\frac{1}{2}})) \int_0^{\underline{t}} \int_{y \in \mathbb{R}} \phi(y)^2 u^{(N)}(s, y)^2 dy ds. \end{aligned}$$

Therefore, we can conjecture that the limit point $M_t^{(e)}(\phi)$ is

$$2\beta^2 \int_0^t \int_{y \in \mathbb{R}} \phi^2(y) u(s, y)^2 ds dy \quad (3.12)$$

if $u^{(N)} \Rightarrow u$ for some $u(s, y)$ in some sense. In the following, we will check that (3.12) is true.

We denote by $\tilde{X}_t^{(N)}$ new measure-valued processes associated to $u^{(N)}(\cdot, \cdot)$, that is for $\phi \in C_b^2(\mathbb{R})$,

$$\tilde{X}_t^{(N)}(\phi) = \int_{\mathbb{R}} \phi(x) u^{(N)}(t, x) dx.$$

Then, it is clear that for $C_b^2(\mathbb{R})$ and for any $K > 0$

$$\limsup_{N \rightarrow \infty} E \left[\sup_{t < K} \left| \tilde{X}_t^{(N)}(\phi) - X_t^{(N)}(\phi) \right| \right] = 0.$$

Thus, $\{\tilde{X}^{(N)} : N \in \mathbb{N}\}$ is C -relative compact in $D([0, \infty), \mathcal{M}_F(\mathbb{R}))$ and there are subsequences which weakly converges to X ., where X . is the one given in (3.11).

We will prove the following lemmas:

Lemma 3.11. *Let X . be a limit point of the sequence $\{X^{(N)} : N \in \mathbb{N}\}$. Then, the measure valued process $\{X_t(\cdot) : 0 \leq t < \infty\}$ is almost surely absolutely continuous for all $t > 0$, that is there exists an adapted Borel-measurable-function-valued process $\{u_t : t > 0\}$ such that*

$$X_t(dx) = u_t(x)dx, \quad \text{for all } t > 0, \quad P\text{-a.s.}$$

Define a sequences of measure valued processes $\{\mu^{(N)}(dx) : N \in \mathbb{N}\}$ by

$$\mu_t^{(N)}(dx) = 2\beta^2 \int_0^t \left(u^{(N)}(s, x)\right)^2 dx ds.$$

Lemma 3.12. *For any $\varepsilon > 0$ and for any $T > 0$, there exists a compact set $K^{\varepsilon, T} \subset \mathbb{R}$ such that*

$$\sup_N P \left(\sup_{t \leq T} \mu_t^{(N)} \left((K^{\varepsilon, T})^c \right) > \varepsilon \right) < \varepsilon.$$

By using Lemma 3.11 and Lemma 3.12, we can complete the proof of Theorem 2.1 as follows:

Complete the proof of Theorem 2.1. We will verify that if $X^{(N_k)}(dx) \Rightarrow u(\cdot, x)dx$ as $N_k \rightarrow \infty$, then

$$\mu_t^{(N_k)}(dx) \Rightarrow \left(2\beta^2 \int_0^t u(s, x)^2 ds \right) dx. \quad (3.13)$$

Actually, $\left\{ \left(\mu_t^{(N)}(\cdot) \right)_{t \in [0, \infty)} : N \in \mathbb{N} \right\}$ are C -relatively compact in $D([0, \infty), \mathcal{M}_F(\mathbb{R}))$ if the conditions in Theorem 3.2 are satisfied. However, we have already checked them in the proof of the tightness of $\{X^{(N)} : N \in \mathbb{N}\}$ and Lemma 3.12. Thus, for any $\phi \in C_b^2(\mathbb{R})$,

$$\mu_t^{(N_k)}(\phi) \Rightarrow \mu_t(\phi) \quad \text{for subsequences } N_k \rightarrow \infty.$$

Also, we may consider this convergence is almost surely by Skorohod representation theorem, that is

$$\lim_{k \rightarrow \infty} \mu_t^{(N_k)}(\phi) = \mu_t(\phi), \quad \text{a.s.} \quad (3.14)$$

Let $G_N(B, m)$ be the distributions of $u^{(N)}(t, x)$ for $B \in \mathcal{B}(\mathbb{R}_{\geq 0} \times \mathbb{R})$ and $m \in [0, \infty)$, that is

$$G_N(B, m) = \left| \left\{ (t, x) \in B : u^{(N)}(t, x) \leq m \right\} \right|,$$

where $|\cdot|$ represents Lebesgue measure on $\mathbb{R}_{\geq 0} \times \mathbb{R}$. Especially,

$$G_N([0, t] \times \mathbb{R}, m) = \frac{2}{N^{\frac{3}{2}}} \# \left\{ (n, x) : n \leq \{0, \dots, \lfloor tN \rfloor\}, x \in \mathbb{Z}, B_{n,x} \leq 2m\sqrt{N} \right\}.$$

Then, the convergence of $u_t^{(N)}(\cdot)$ in (3.14) is equivalent to the convergence of the distributions $G_N(\cdot, \cdot)$.

Let $\mu_t^{(M,N)}(\cdot)$ be the truncated measure of $\mu_t^{(N)}(\cdot)$ for $M > 0$, that is

$$\mu_t^{(M,N)}(dx) = \left(2\beta^2 \int_0^t \left(u^{(N)}(s, x) \wedge M \right)^2 ds \right) dx.$$

Then, it is clear that for any bounded function $C_{b,+}^2(\mathbb{R})$

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}} \phi(x) \left(u^{(N)}(s, x) \wedge M \right)^2 dx ds \\ &= 2 \int_0^t \int_{\mathbb{R}} \int_0^M \phi(x) m^2 G_N(ds dx dm) \\ &+ 2 \int_0^t \int_{\mathbb{R}} \int_M^\infty \mathbf{1}_{\{u^{(N)}(s,x) > M\}} \phi(x) M^2 G_N(ds dx dm). \end{aligned}$$

The last term converges to 0 in probability as $N \rightarrow \infty$ and then $M \rightarrow \infty$. Indeed, we have that

$$\begin{aligned} 0 &\leq \int_0^t \int_{\mathbb{R}} \int_M^\infty \mathbf{1}_{\{u^{(N)}(s,x) > M\}} \phi(x) M^2 G_N(ds dx dm) \\ &\leq C(\phi) \frac{\left(B_{n,x}^{(N)} \right)^2}{N^{\frac{3}{2}}} \# \left\{ (n, x) : n \leq \{0, \dots, \lfloor tN \rfloor\}, x \in \mathbb{Z}, B_{n,x} \geq 2M\sqrt{N} \right\}, \end{aligned}$$

and the last term converges to 0 in probability by Lemma 3.9. Also, as $N_k \rightarrow \infty$

$\int_0^t \int_{\mathbb{R}} \int_0^M \phi(x) m^2 G_{N_k}(ds dx dm)$ converges almost surely to

$$\int_0^t \int_{\mathbb{R}} \int_0^M \phi(x) m^2 G(ds dx dm) = \int_0^t \int_{\mathbb{R}} \phi(x) u(s, x)^2 \mathbf{1}_{\{u(t, x) \leq M\}} dx ds,$$

where $G(\cdot, \cdot, \cdot)$ is the distribution of $u(t, x)$. Thus, we have that for any $\phi \in C_{b,+}^2(\mathbb{R})$

$$\begin{aligned} \int_0^t \int_{\mathbb{R}} \phi(x) u(s, x)^2 dx ds &= \lim_{M \rightarrow \infty} \lim_{N_k \rightarrow \infty} \int_0^t \int_{\mathbb{R}} \int_0^M \phi(x) m^2 G_{N_k}(ds dx dm) \\ &\leq \lim_{M \rightarrow \infty} \lim_{N_k \rightarrow \infty} \int_0^t \int_{\mathbb{R}} \phi(x) \left(u^{(N_k)}(t, x) \wedge M \right)^2 dx ds \\ &\leq \mu_t(\phi), \quad \text{a.s.} \end{aligned}$$

Also, we know that for bounded function $\phi \in C_{b,+}^2(\mathbb{R})$, for any $t > 0$ and for any $\varepsilon > 0$

$$\begin{aligned} & \lim_{M \rightarrow \infty} \sup_N P \left(\left| \int_0^t \int_{\mathbb{R}} \phi(x) \left(\left(u^{(N)}(s, x) \right)^2 - \left(u^{(N)}(s, x) \wedge M \right)^2 \right) dx ds \right| > \varepsilon \right) \\ &\leq \lim_{M \rightarrow \infty} \sup_N P \left(\left| \int_0^t \int_{\mathbb{R}} \int_M^\infty \phi(x) m^2 G_N(ds dx dm) \right| > \varepsilon \right) \\ &= 0, \end{aligned}$$

by Lemma 3.9. Thus, for any bounded function $\phi \in C_{b,+}^2(\mathbb{R})$

$$\begin{aligned}\mu_t(\phi) &= \lim_{N_k \rightarrow \infty} 2\beta^2 \int_0^t \int_{\mathbb{R}} \phi(x) \left(u^{(N_k)}(t, x)\right)^2 dx ds \\ &\leq 2\beta^2 \int_0^t \int_{\mathbb{R}} \phi(x) u(t, x)^2 dx ds, \quad \text{in probability.}\end{aligned}$$

This is true for $\phi \in C_b^2(\mathbb{R})$. Thus, we have proved (3.13). \square

Proof of Lemma 3.12. First, we remark that $M_{\underline{t}}^{(e,N)}(\phi)$ is an $\mathcal{F}_{\underline{t}N}^{(N)}$ -martingale even if $\phi(x) = \mathbf{1}_K(x)$ for Borel measurable set K . Then,

$$\left\langle M^{(e,N)}(K^c) \right\rangle_{\underline{t}} = \frac{1}{N} \sum_{\underline{s} < \underline{t}} \sum_{x \in K^c N^{\frac{1}{2}}} \frac{\left(\beta B_{\underline{s}N,x}^{(N)}\right)^2}{N^{\frac{3}{2}}} = 2\beta^2(1 + \mathcal{O}(N^{-\frac{1}{2}}))\mu_t(K^c)$$

is an increasing process. Thus, we have that

$$\begin{aligned}P\left(\sup_{t \leq T} \mu_t(K^c) > \varepsilon\right) &\leq P\left(3 \sup_{t \leq T} \left\langle M^{(e,N)}(K^c) \right\rangle_{\underline{t}} > \varepsilon\right) \\ &\leq \varepsilon^{-1} E \left[\frac{3}{N} \sum_{\underline{s} < T} \sum_{x \in K^c N^{\frac{1}{2}}} \frac{\left(\beta B_{\underline{s}N,x}^{(N)}\right)^2}{N^{\frac{3}{2}}} \right] \\ &\leq \varepsilon^{-1} C \sum_{\underline{s} < T} \sum_{x \in K^c N^{\frac{1}{2}}} \frac{\beta^2(s \vee 1)^2}{N\sqrt{s}} P_Y(Y_{\underline{s}N} = x) \\ &\leq \varepsilon^{-1} C \beta^2 \sqrt{T} \left(\sup_{\underline{s} < T} P_Y(Y_{\underline{s}N} \in K^c N^{\frac{1}{2}}) \right) \\ &\leq \varepsilon,\end{aligned}$$

by taking K^c as a compact set in \mathbb{R} such that $C\beta^2\sqrt{K} \sup_{\underline{s} < T} P_Y(Y_{\underline{s}N} \in K^c N^{\frac{1}{2}}) \leq \varepsilon^2$, where we used Lemma 5.1 in the third inequality. \square

In the rest of this section, we will prove Lemma 3.11.

For $\psi \in C_b^{1,2}([0, \infty) \times \mathbb{R}, \mathbb{R})$, we define

$$X_t^{(N)}(\psi_t) = \sum_{\mathbf{x} \sim \underline{t}} \frac{\psi\left(t, \frac{Y_{\underline{t}N}^{\mathbf{x}}}{N^{\frac{1}{2}}}\right)}{N}, \quad (3.15)$$

where $\psi_t(x) = \psi(t, x)$. Also, we have the following equation

$$\begin{aligned}
& X_{\underline{t} + \frac{1}{N}}^{(N)}(\psi_{\underline{t} + \frac{1}{N}}) - X_{\underline{t}}^{(N)}(\psi_{\underline{t}}) \\
&= \sum_{\mathbf{x} \sim \underline{t}} \frac{\psi\left(\underline{t} + \frac{1}{N}, \frac{Y_{\underline{t}N+1}^{\mathbf{x}}}{N^{\frac{1}{2}}}\right)}{N} \left(V^{\mathbf{x}} - 1 - \frac{\beta \xi\left(\underline{t}N, Y_{\underline{t}N}^{\mathbf{x}}\right)}{N^{\frac{1}{4}}} \right) \\
&\quad + \sum_{\mathbf{x} \sim \underline{t}} \frac{\psi\left(\underline{t} + \frac{1}{N}, \frac{Y_{\underline{t}N+1}^{\mathbf{x}}}{N^{\frac{1}{2}}}\right)}{N} \frac{\beta \xi\left(\underline{t}N, Y_{\underline{t}N}^{\mathbf{x}}\right)}{N^{\frac{1}{4}}} \\
&\quad + \sum_{\mathbf{x} \sim \underline{t}} \frac{2\psi\left(\underline{t} + \frac{1}{N}, \frac{Y_{\underline{t}N+1}^{\mathbf{x}}}{N^{\frac{1}{2}}}\right) - \psi\left(\underline{t} + \frac{1}{N}, \frac{Y_{\underline{t}N}^{\mathbf{x}}}{N^{\frac{1}{2}}}\right) - \psi\left(\underline{t} + 1/N, \frac{Y_{\underline{t}N}^{\mathbf{x}} - 1}{N^{\frac{1}{2}}}\right)}{2N} \\
&\quad + \sum_{\mathbf{x} \sim \underline{t}} \frac{\psi\left(\underline{t} + \frac{1}{N}, \frac{Y_{\underline{t}N+1}^{\mathbf{x}}}{N^{\frac{1}{2}}}\right) + \psi\left(\underline{t} + \frac{1}{N}, \frac{Y_{\underline{t}N}^{\mathbf{x}} - 1}{N^{\frac{1}{2}}}\right) - 2\psi\left(\underline{t}, \frac{Y_{\underline{t}N}^{\mathbf{x}}}{N^{\frac{1}{2}}}\right)}{2N} \\
&=: \Delta M_{\underline{t} + \frac{1}{N}}^{(b,N)}(\psi_{\underline{t} + \frac{1}{N}}) + \Delta M_{\underline{t} + \frac{1}{N}}^{(e,N)}(\psi_{\underline{t} + \frac{1}{N}}) \\
&\quad + \Delta M_{\underline{t} + \frac{1}{N}}^{(s,N)}(\psi_{\underline{t} + \frac{1}{N}}) + \Delta C_{\underline{t} + \frac{1}{N}}^{(N)}(\psi_{\underline{t} + \frac{1}{N}}).
\end{aligned}$$

For $i = b, e, s$, $M_t^{(i,N)}(\psi_t)$ which are the sums of $\Delta M_t^{(i,N)}(\psi_t)$ up to t are martingales with respect to $\mathcal{F}_{\underline{t}N}^{(N)}$ as well as $M_t^{(i,N)}(\phi)$ are.

We take ψ as the shift of $\frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right)$;

$$\psi_t^x(y) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(y-x)^2}{2t}\right).$$

Then, we have that for $\varepsilon, \varepsilon' > 0$ and $t \geq \eta > 0$

$$\begin{aligned}
& E \left[\left(X_t^{(N)}(\psi_{\varepsilon}^x) - X_t^{(N)}(\psi_{\varepsilon'}^x) \right)^2 \right] \\
& \leq \sum_{\underline{s} \leq \underline{t}} E \left[\left(\Delta M_{\underline{s}}^{(b,N)} \left(\psi_{t+\varepsilon-\underline{s}}^x - \psi_{t+\varepsilon'-\underline{s}}^x \right) \right)^2 \right] \quad (\text{Mb}) \\
& \quad + \sum_{\underline{s} \leq \underline{t}} E \left[\left(\Delta M_{\underline{s}}^{(e,N)} \left(\psi_{t+\varepsilon-\underline{s}}^x - \psi_{t+\varepsilon'-\underline{s}}^x \right) \right)^2 \right] \quad (\text{Me}) \\
& \quad + \sum_{\underline{s} \leq \underline{t}} E \left[\left(\Delta M_{\underline{s}}^{(s,N)} \left(\psi_{t+\varepsilon-\underline{s}}^x - \psi_{t+\varepsilon'-\underline{s}}^x \right) \right)^2 \right] \quad (\text{Ms}) \\
& \quad + E \left[\left(\sum_{\underline{s} \leq \underline{t}} \Delta C_{\underline{s}}^{(N)} \left(\psi_{t+\varepsilon-\underline{s}}^x - \psi_{t+\varepsilon'-\underline{s}}^x \right) \right)^2 \right] \quad (\text{C}) \\
& \quad + \left(\psi_{t+\varepsilon}^x(0) - \psi_{t+\varepsilon'}^x(0) \right)^2 \quad (\text{Initial term}) \\
& \quad + E \left[\left(\sum_{\mathbf{x} \sim \underline{t}} \frac{\left(\psi_{\varepsilon}^x - \psi_{t+\varepsilon-\underline{t}}^x - \psi_{\varepsilon'}^x + \psi_{t+\varepsilon'-\underline{t}}^x \right) \left(\frac{Y_{\underline{t}N}^{\mathbf{x}}}{N^{\frac{1}{2}}} \right)}{2\sqrt{N}} \right)^2 \right] \quad (\text{Error term})
\end{aligned}$$

Clearly, for fixed $\varepsilon > 0$, $\sup_y |\psi_\varepsilon^x(y) - \psi_{t+\varepsilon-\underline{t}}^x(y)| \leq \frac{C(\varepsilon)}{N}$. So (Error term) is bounded above by

$$E \left[\left(X_{\underline{t}}^{(N)} \left(\frac{C(\varepsilon) + C(\varepsilon')}{N} \right) \right)^2 \right] \rightarrow 0, \text{ as } N \rightarrow \infty.$$

Also,

$$(\text{Initial term}) \leq (\varepsilon - \varepsilon')^2 ((t + \varepsilon) \wedge (t + \varepsilon'))^{-3},$$

where we have used [23, Lemma III 4.5 (a)], that is for $0 \leq \delta \leq p$,

$$|\psi_{t+\varepsilon}^x(y) - \psi_t^x(y)|^p \leq (\varepsilon t^{-3/2})^\delta \left((\psi_{t+\varepsilon}^x(y))^{p-\delta} + (\psi_t^x(y))^{p-\delta} \right) \quad (3.16)$$

for all $x, y \in \mathbb{R}$, $t > 0$, and $\varepsilon > 0$.

Lemma 3.13. *For $\varepsilon, \varepsilon' > 0$ and $t \geq \eta > 0$,*

$$\lim_{N \rightarrow \infty} E \left[\left(\sum_{\underline{s} \leq \underline{t}} \Delta C_{\underline{s}}^{(N)} \left(\psi_{t+\varepsilon-\underline{s}}^x - \psi_{t+\varepsilon'-\underline{s}}^x \right) \right)^2 \right] = 0.$$

Proof.

$$\begin{aligned} & \Delta C_{\underline{s}}^{(N)} \left(\psi_{t+\varepsilon-\underline{s}}^x \right) \\ &= \sum_{\mathbf{x} \sim \underline{s}} \frac{\psi_{t+\varepsilon-\underline{s}-\frac{1}{N}}^x \left(\frac{Y_{\underline{t}N}^{\mathbf{x}}+1}{N^{\frac{1}{2}}} \right) + \psi_{t+\varepsilon-\underline{s}-\frac{1}{N}}^x \left(\frac{Y_{\underline{t}N}^{\mathbf{x}}-1}{N^{\frac{1}{2}}} \right) - \psi_{t+\varepsilon-\underline{s}}^x \left(\frac{Y_{\underline{t}N}^{\mathbf{x}}+1}{N^{\frac{1}{2}}} \right) - \psi_{t+\varepsilon-\underline{s}}^x \left(\frac{Y_{\underline{t}N}^{\mathbf{x}}-1}{N^{\frac{1}{2}}} \right)}{2N} \\ &+ \sum_{\mathbf{x} \sim \underline{s}} \frac{\psi_{t+\varepsilon-\underline{s}}^x \left(\frac{Y_{\underline{t}N}^{\mathbf{x}}+1}{N^{\frac{1}{2}}} \right) + \psi_{t+\varepsilon-\underline{s}}^x \left(\frac{Y_{\underline{t}N}^{\mathbf{x}}-1}{N^{\frac{1}{2}}} \right) - 2\psi_{t+\varepsilon-\underline{s}}^x \left(\frac{Y_{\underline{t}N}^{\mathbf{x}}}{N^{\frac{1}{2}}} \right)}{2N} \\ &\leq \sum_{\mathbf{x} \sim \underline{s}} \frac{1}{N^2} \left(\left| \frac{\partial \psi^x \left(t + \varepsilon - s, \frac{Y_{\underline{s}N}^{\mathbf{x}}}{N^{\frac{1}{2}}} \right)}{\partial s} \right|_{s=\underline{s}} + \mathcal{O}(N^{-\frac{1}{2}}) \right) \\ &+ \sum_{\mathbf{x} \sim \underline{s}} \frac{1}{N^2} \left(\left| \frac{\partial^2 \psi^x(t + \varepsilon - \underline{s}, y)}{2\partial y^2} \right|_{y=\frac{Y_{\underline{s}N}^{\mathbf{x}}}{N^{\frac{1}{2}}}} + \mathcal{O}(N^{-\frac{1}{2}}) \right) \end{aligned}$$

Since $\frac{\partial \psi^x(t+\varepsilon-s, y)}{\partial s} + \frac{\partial^2 \psi^x(t+\varepsilon-s, y)}{2\partial y^2} = 0$, the last equation is bounded above by

$$\left| \Delta C_{\underline{s}}^{(N)} \left(\psi_{t+\varepsilon-\underline{s}}^x \right) \right| \leq C(\varepsilon, \eta) \frac{X_{\underline{s}}^{(N)}(1)}{N^{\frac{3}{2}}}.$$

Thus,

$$\begin{aligned} E \left[\left(\sum_{\underline{s} \leq \underline{t}} \Delta C_{\underline{s}}^{(N)} \left(\psi_{t+\varepsilon-\underline{s}}^x - \psi_{t+\varepsilon'-\underline{s}}^x \right) \right)^2 \right] &\leq E \left[(C(\varepsilon, \eta) + C(\varepsilon', \eta))^2 \sup_{\underline{s} \leq \underline{t}} \left(\frac{X_{\underline{s}}^{(N)}(1)}{N^{\frac{1}{2}}} \right)^2 \right] \\ &\rightarrow 0 \text{ as } N \rightarrow \infty. \end{aligned}$$

Indeed, for each N , $X_{\underline{s}}^{(N)}(1)$ is a martingale so that by L^2 -maximum inequality and by Lemma 3.7,

$$\sup_N E \left[\sup_{\underline{s} \leq \underline{t}} \left(X_{\underline{s}}^{(N)}(1) \right)^2 \right] \leq 4 \sup_N E \left[\left\langle X^{(N)}(1) \right\rangle_{\underline{t}} \right] < \infty.$$

□

Thus, we have by Fatou's lemma that

$$E \left[\left(X_t(\psi_\varepsilon^x) - X_t(\psi_{\varepsilon'}^x) \right)^2 \right] \leq \liminf_{N \rightarrow \infty} ((Mb) + (Me) + (Ms)).$$

Hereafter, we will see the right hand side .

Lemma 3.14. *Suppose $\varepsilon > \varepsilon' > 0$, $t \geq \eta > 0$, and $0 < \delta < \frac{1}{2}$. Then, for any $x \in \mathbb{R}$*

$$\liminf_{N \rightarrow \infty} (Mb) \leq C_\delta (\varepsilon - \varepsilon')^\delta (t + \varepsilon')^{-\delta}.$$

Proof. By Lemma 3.6, we have that for $\varepsilon > \varepsilon' > 0$, for $t \geq \eta > 0$, and for $0 < \delta < \frac{1}{2}$

(Mb)

$$\begin{aligned} &= \left(1 - \frac{1}{N^{\frac{1}{2}}} \right) E \left[\sum_{\underline{s} \leq \underline{t}} \sum_{z \in \mathbb{Z}} \frac{\left(\psi_{t+\varepsilon-\underline{s}}^x \left(\frac{z}{N^{\frac{1}{2}}} \right) - \psi_{t+\varepsilon'-\underline{s}}^x \left(\frac{z}{N^{\frac{1}{2}}} \right) \right)^2}{N^2} B_{\underline{s}N, z}^{(N)} \right] \\ &\leq E_Y \left[\sum_{\underline{s} \leq \underline{t}} \frac{\left(\psi_{t+\varepsilon-\underline{s}}^x \left(\frac{Y_{\underline{s}N}}{N^{\frac{1}{2}}} \right) - \psi_{t+\varepsilon'-\underline{s}}^x \left(\frac{Y_{\underline{s}N}}{N^{\frac{1}{2}}} \right) \right)^2}{N} \right], \end{aligned}$$

and it follows from (3.16) that

$$\leq \int_0^t E_Y \left[\left(\frac{\varepsilon - \varepsilon'}{(t + \varepsilon' - s)^{\frac{3}{2}}} \right)^\delta \left(\left(\psi_{t+\varepsilon-s}^x \left(\frac{Y_{sN}}{N^{\frac{1}{2}}} \right) \right)^{2-\delta} + \left(\psi_{t+\varepsilon'-s}^x \left(\frac{Y_{sN}}{N^{\frac{1}{2}}} \right) \right)^{2-\delta} \right) \right] ds.$$

Thus, we have from invariance principle that

$$\begin{aligned} &\liminf_{N \rightarrow \infty} (Mb) \\ &\leq \int_0^t \int_{\mathbb{R}} \left(\frac{\varepsilon - \varepsilon'}{(t + \varepsilon' - s)^{\frac{3}{2}}} \right)^\delta \left(\left(\psi_{t+\varepsilon-s}^x(y) \right)^{2-\delta} + \left(\psi_{t+\varepsilon'-s}^x(y) \right)^{2-\delta} \right) \psi_s^0(y) dy ds \\ &\leq (\varepsilon - \varepsilon')^\delta \int_0^t (t + \varepsilon' - s)^{-\frac{3\delta}{2}} (2 - \delta)^{-\frac{1}{2}} \left((t + \varepsilon - s)^{\frac{\delta-1}{2}} \left(\frac{2 - \delta}{t + \varepsilon + (1 - \delta)s} \right)^{\frac{1}{2}} \right) ds \\ &\quad + (\varepsilon - \varepsilon')^\delta \int_0^t (t + \varepsilon' - s)^{-\frac{3\delta}{2}} (2 - \delta)^{-\frac{1}{2}} \left((t + \varepsilon' - s)^{\frac{\delta-1}{2}} \left(\frac{2 - \delta}{t + \varepsilon' + (1 - \delta)s} \right)^{\frac{1}{2}} \right) ds \\ &\leq C_\delta (\varepsilon - \varepsilon')^\delta (t + \varepsilon')^{-\frac{1}{2}} \int_0^t (t + \varepsilon' - s)^{-\frac{1}{2} - \delta} ds \leq C_\delta (\varepsilon - \varepsilon')^\delta (t + \varepsilon')^{-\frac{1}{2}} (t + \varepsilon')^{\frac{1}{2} - \delta}, \end{aligned}$$

where we have used the fact that $\int_{\mathbb{R}} \psi_s^x(y) \psi_t^0(y) dy = \psi_{t+s}^0(x)$ in the second inequality. \square

Lemma 3.15. *For all $x \in \mathbb{R}$, $\varepsilon > \varepsilon' > 0$, and $t \geq \eta > 0$, we have*

$$\lim_{N \rightarrow \infty} (Ms) = 0.$$

Proof. The proof is the same as the proof of Lemma 3.5. \square

Lemma 3.16. *Suppose $\varepsilon > \varepsilon' > 0$, $t \geq \eta > 0$, and $0 < \delta < \frac{1}{2}$. Then, for any $x \in \mathbb{R}$*

$$\liminf_{N \rightarrow \infty} (Me) \leq C(\delta) \beta^2 (t \vee 1)^2 (t + \varepsilon')^{-\frac{1}{2} - \delta} (\varepsilon - \varepsilon')^\delta.$$

Proof. By Lemma 5.2, we have that

$$\begin{aligned} & (\text{Me}) \\ & \leq \beta^2 E \left[\sum_{\underline{s} \leq \underline{t}} \sum_{z \in \mathbb{Z}} \frac{\left(\psi_{t+\varepsilon-\underline{s}}^x \left(\frac{z}{N^{\frac{1}{2}}} \right) - \psi_{t+\varepsilon'-\underline{s}}^x \left(\frac{z}{N^{\frac{1}{2}}} \right) \right)^2 \left(B_{\underline{s}N, z}^{(N)} \right)^2}{N N^{\frac{3}{2}}} \right] \\ & \leq \beta^2 \sum_{\underline{s} \leq \underline{t}} \sum_{z \in \mathbb{Z}} \frac{C(\underline{s} \vee 1)^2}{N \sqrt{\underline{s}}} \left(\psi_{t+\varepsilon-\underline{s}}^x \left(\frac{z}{N^{\frac{1}{2}}} \right) - \psi_{t+\varepsilon'-\underline{s}}^x \left(\frac{z}{N^{\frac{1}{2}}} \right) \right)^2 P(Y_{\underline{s}N} = z) \\ & \leq C \beta^2 (t \vee 1)^2 \int_0^t \sum_{z \in \mathbb{Z}} \frac{1}{\sqrt{s}} \left(\psi_{t+\varepsilon-s}^x \left(\frac{z}{N^{\frac{1}{2}}} \right) - \psi_{t+\varepsilon'-s}^x \left(\frac{z}{N^{\frac{1}{2}}} \right) \right)^2 P(Y_{\underline{s}N} = z) ds, \end{aligned}$$

where we have used Lemma 5.1 in the third inequality. Let $0 < \eta' < t$. Then, we obtain by the similar argument in the proof of Lemma 3.14 that

$$\begin{aligned} & \liminf_{N \rightarrow \infty} (\text{Mb}) \\ & \leq C \beta^2 (t \vee 1)^2 \left(\int_{\eta'}^t \int_{\mathbb{R}} \frac{1}{\sqrt{s}} \left(\psi_{t+\varepsilon-s}^x(y) - \psi_{t+\varepsilon'-s}^x(y) \right)^2 \psi_s^0(y) dy ds \right. \\ & \quad \left. + \int_0^{\eta'} \frac{\sup_y \left(\psi_{t+\varepsilon-s}^x(y) - \psi_{t+\varepsilon'-s}^x(y) \right)^2}{\sqrt{s}} ds \right) \\ & \leq C \beta^2 (t \vee 1)^2 \int_{\eta'}^t \int_{\mathbb{R}} \left(\frac{\varepsilon - \varepsilon'}{(t + \varepsilon' - s)^{\frac{3}{2}}} \right)^\delta \left(\left(\psi_{t+\varepsilon-s}^x(y) \right)^{2-\delta} + \left(\psi_{t+\varepsilon'-s}^x(y) \right)^{2-\delta} \right) \frac{\psi_s^0(y)}{\sqrt{s}} dy ds \\ & \quad + C \beta^2 (t \vee 1)^2 \int_0^{\eta'} \left(\frac{\varepsilon - \varepsilon'}{(t + \varepsilon' - s)^{\frac{3}{2}}} \right)^\delta s^{-\frac{1}{2}} \left((t + \varepsilon - s)^{\frac{2-\delta}{2}} + (t + \varepsilon' - s)^{\frac{2-\delta}{2}} \right) ds \\ & \leq C(\delta) \beta^2 \frac{(t \vee 1)^2}{(t + \varepsilon')^{\frac{1}{2}}} (\varepsilon - \varepsilon')^\delta \int_0^{t+\varepsilon'} s^{-\frac{1}{2}} (t + \varepsilon' - s)^{-\frac{1}{2} - \delta} ds \\ & \quad + C(\delta) \beta^2 (t \vee 1)^2 (\varepsilon - \varepsilon')^\delta \int_0^{\eta'} s^{-\frac{1}{2}} \frac{(t + \varepsilon' - s)^{\frac{2-\delta}{2}} + (t + \varepsilon - s)^{\frac{2-\delta}{2}}}{(t + \varepsilon' - s)^\delta} ds \\ & \leq C(\delta) \beta^2 (t \vee 1)^2 (\varepsilon - \varepsilon')^\delta \left((t + \varepsilon')^{-\frac{1}{2} - \delta} B \left(\frac{1}{2}, \frac{1}{2} - \delta \right) + \eta'^{\frac{1}{2}} (t + \varepsilon)^{\frac{2-\delta}{2}} (t + \varepsilon' - \eta')^{-\delta} \right). \end{aligned}$$

Since $\eta' > 0$ is arbitrary, we have that

$$\liminf_{N_k \rightarrow \infty} (\text{Me}) \leq C(\delta) \beta^2 (t \vee 1)^2 (t + \varepsilon')^{-\frac{1}{2} - \delta} (\varepsilon - \varepsilon')^\delta.$$

□

Thus, we have that

$$\lim_{\varepsilon, \varepsilon' \rightarrow 0} \sup_{x \in \mathbb{R}, t \geq \eta} E \left[(X_t(\psi_\varepsilon^x) - X_t(\psi_{\varepsilon'}^x))^2 \right] = 0, \quad \text{for any } \eta > 0.$$

By Skorohod representation theorem, we may assume that $X^{(N_k)}$ and X are defined on a common probability space and $X^{(N_k)} \rightarrow X$ in $D([0, \infty), \mathcal{M}_F(\mathbb{R}))$ a.s.. Then, from the above arguments, we have that

$$X_t(\psi_\varepsilon^x) = X_0(\psi_{t+\varepsilon}^x) + \tilde{M}_t(\psi_{t+\varepsilon-}^x) \quad (3.17)$$

for a certain continuous L^2 -bounded martingale $\tilde{M}_t(\psi_{t+\varepsilon-}^x)$, where the martingale property of $\tilde{M}_t(\psi_{t+\varepsilon-}^x)$ is obtained by the same argument as the proof of Lemma 3.1. Also, we take L^2 -limit in (3.17) as $\varepsilon \rightarrow 0$ and choose $\varepsilon_n \rightarrow 0$ so that for any t and $x \in \mathbb{R}$,

$$\lim_{n \rightarrow \infty} X_t(\psi_{\varepsilon_n}^x) = X_0(\psi_t^x) + \tilde{M}_t(\psi_{t-}^x) \quad \text{a.s. and in } L^2. \quad (3.18)$$

We define $u(t, x) = \lim_{\varepsilon_n \rightarrow 0} X_t(\psi_{\varepsilon_n}^x)$ for all $t > 0$, $x \in \mathbb{R}$. Standard differential theory shows that for each $t > 0$ with probability 1,

$$X_t(dx) = u(t, x)dx + X_t^s(dx),$$

where X_t^s is a random measure such that $X_t^s(dx) \perp dx$. Also, (3.18) implies that

$$E \left[\int_{\mathbb{R}} u(t, x) dx \right] = \int_{\mathbb{R}} X_0(\psi_t^x) dx = 1 = E[X_t(1)].$$

Thus, $E[X_t^s(1)] = 0$ and

$$X_t(dx) = u(t, x)dx, \quad \text{a.s. for all } t > 0.$$

Therefore, we complete the proof of Lemma 3.11 and also Theorem 2.1.

4 Proof of Theorem 2.2

In this section, we will prove the weak uniqueness of $u_t(x)$ where $u_t(x) = u(t, x)$ is a solution of the SPDE

$$\frac{\partial}{\partial t} u(t, x) = \frac{1}{2} \Delta u(t, x) + \sqrt{au + bu^2} \dot{W}(t, x), \quad u(0, x) = \psi(x) \in C_{rap}^+(\mathbb{R}), \quad (4.1)$$

for $a, b > 0$. The proof is essentially based on the moment duality argument developed in [1]. If we can find the “moment dual process”, then $E[\prod_{i=1}^N u(t, x_i)]$

is independent of the choice of the solutions for any $t > 0$ and $x_i \in \mathbb{R}$. Also, it implies that for $\phi \in C_c^+(\mathbb{R})$

$$E[\langle u_t, \phi \rangle^N] = \int_{\mathbb{R}^N} \left(\prod_{i=1}^N \phi(x_i) \right) E \left[\prod_{j=1}^N u(t, x_j) \right] d\mathbf{x}$$

is independent of the choice of solutions and so is $E[\exp(-\langle u_t, \phi \rangle)]$ if they satisfy the Carleman's condition. Also, this independence is true for $\phi \in C_b^+(\mathbb{R})$. Thus, the weak uniqueness for the solutions of (4.1) follows.

Dual particle system. Our dual process is the same as the one defined in [1]. We consider the following particle systems in \mathbb{R} .

Let $I_0 = \{1, 2, \dots, N\}$ be the initial set of labels of particles. Also, I_t is the set of labels of particles at time t . In our case, we can take I_t be a subset of $\{1, \dots, N\}$ as below. At time 0, there are N particles located site x_1, \dots, x_N . During their lifetimes, the particle performs independent Brownian motions. The process evolves as follows. We denote X_t^β by the position of the particle β at time t if β exists.

Coalescence with spatial correlation. Two particles β and γ collide into the one particle $\beta \cup \gamma$ (independently of other pairs) at the rate $\frac{a}{2} dL_t^{\beta, \gamma}$, where $L_t^{\beta, \gamma}$ denotes the local time of the process $X_t^\beta - X_t^\gamma$ at 0.

For convenience, we shall take a left continuous version. We denote by P_X , and E_X the probability measure and the expectation associated with $\{X_t^\beta : \beta \in I_t, 0 \leq t\}$.

We set

$$L_t = \frac{1}{2} \int_0^t \sum_{\beta \in I_s} \sum_{\gamma \in I_s \setminus \{\beta\}} dL_s^{\beta, \gamma},$$

the total local time accrued by all pairs of particles at time t . The reader may refer to [1] for more detail.

Theorem 4.1. *Suppose that u is a solution of (4.1). Then, the following duality relation holds for any $T \geq 0$:*

$$E \left[\prod_{i=1}^N u_T(x_i) \right] = E_X \left[\prod_{\gamma \in I_T} \psi(X_T^\gamma) \exp \left(\left(\frac{a+b}{2} \right) L_T \right) \right].$$

Before discussing the proof of Theorem 4.1, we check the boundedness of the moment.

Lemma 4.2. *(See [16, Proposition 4.2].) Assume $p \geq 2$. Let u be any solutions of (4.1). Then for each $T > 0$, we have that*

$$\sup_{t \leq T, x \in \mathbb{R}} E[u(t, x)^p] < \infty.$$

Now, we will start the construction of the dual process. Let $\nu = \frac{a+b}{2}$ and $\sigma(z) = az + bz^2$. For $\ell > 0$, we set

$$\rho_\ell = \inf\{t \geq 0 : L_t \geq \ell\}.$$

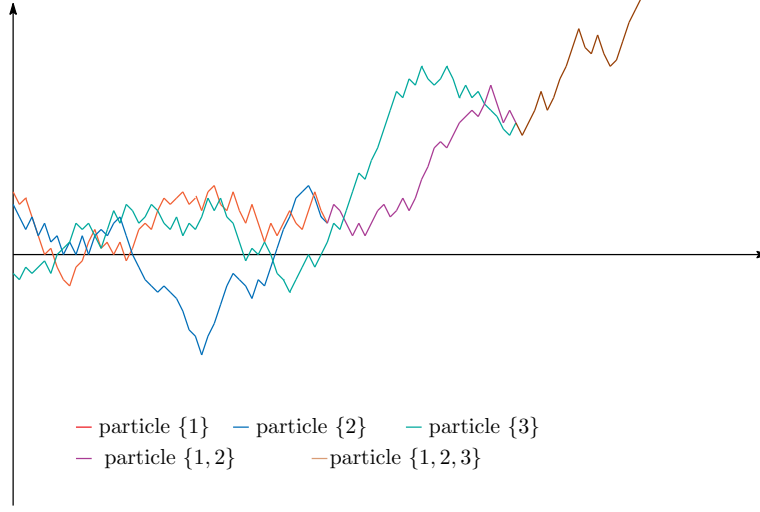


Figure 1: An image of the dual process.

We truncate the dual process X as follows. Let $\{W_t^\gamma : \gamma \subset \{1, 2, \dots, N\}\}$ be an independent collection of Brownian motions. Define

$$\begin{aligned}\tilde{X}_t^\gamma &= \begin{cases} X_t^\gamma, & \text{if } t < \rho_\ell \\ X_{\rho_\ell}^\gamma + W_{t-\rho_\ell}^\gamma, & \text{if } t \geq \rho_\ell \text{ and } \gamma \in I_{\rho_\ell}; \end{cases} \\ \tilde{I}_t &= \{\gamma : \tilde{X}_t^\gamma \text{ are alive at time } t\}; \\ \tilde{L}_t^{\beta, \gamma} &= \text{the local time of } \tilde{X}_t^\beta - \tilde{X}_t^\gamma \text{ at } 0 \text{ up to time } t \wedge \rho_\ell. \end{aligned}$$

Proposition 4.3. *We that for all $T \geq 0$,*

$$E \left[\prod_{i=1}^N u_T(x_i) \right] = E_X \left[\prod_{\gamma \in \tilde{I}_T} \phi(\tilde{X}_T^\gamma) \exp(\nu \tilde{L}_T) \right] + \mathcal{E},$$

where the error term \mathcal{E} is give by

$$\frac{1}{4} E \left[E_X \left[\int_{\rho_\ell}^T \exp(\nu \tilde{L}_t) \sum_{\beta, \gamma \in \tilde{I}_t; \beta \neq \gamma} \left(\prod_{\alpha \in \tilde{I}_t \setminus \{\beta, \gamma\}} u_{T-t}(\tilde{X}_t^\alpha) \right) \sigma(u_{T-t}(\tilde{X}_t^\gamma)) d\tilde{L}_t^{\beta, \gamma} \right] \right].$$

Proof. For any function $f : \mathbb{R} \rightarrow \mathbb{R}$, let f^ε be the convolution of f and p_ε , where $p_\varepsilon(x) = \frac{1}{\sqrt{2\pi\varepsilon}} \exp(-\frac{x^2}{2\varepsilon})$. Then, the process $u^\varepsilon(t, x)$ is a semimartingale and we know that for any $t \geq 0$,

$$u^\varepsilon(t, x) = \phi^\varepsilon(x) + \int_0^t \frac{1}{2} \Delta u^\varepsilon(s, x) ds + \int_0^t \int \sqrt{\sigma(u(s, y))} p_\varepsilon(x - y) dW_{s, y}.$$

Applying Itô's formula, and taking expectations, we have that

$$\begin{aligned}
& E \left[\prod_{i=1}^N u^\varepsilon(t, x_i) \right] - E \left[\prod_{i=1}^N \phi^\varepsilon(x_i) \right] \\
&= E \left[\int_0^t \sum_{i=1}^N \left(\prod_{j=1, j \neq i}^N u^\varepsilon(s, x_j) \right) \left(\frac{1}{2} \Delta u^\varepsilon(s, x_i) \right) ds \right] \\
&\quad + \frac{1}{2} E \left[\int_0^t \sum_{\substack{i,j=1, \\ i \neq j}}^N \left(\prod_{\substack{k=1, \\ k \neq i,j}}^N u^\varepsilon(s, x_k) \right) p_\varepsilon(y - x_i) p_\varepsilon(y - x_j) \sigma(u(s, y)) dy ds \right].
\end{aligned}$$

We replace x_i 's by random set of points, $\{\tilde{X}_r^\beta : \beta \in \tilde{I}_r\}$ which are independent $\{u(s, x) : (s, x) \in [0, \infty) \times \mathbb{R}\}$, and multiply it by $\exp(\nu \tilde{L}_r)$. Then,

$$\begin{aligned}
& E_X \left[E \left[\prod_{\beta \in \tilde{I}_r} u^\varepsilon(s, \tilde{X}_r^\beta) \exp(\nu \tilde{L}_r) \right] \right] - E_X \left[E \left[\prod_{\beta \in \tilde{I}_r} \phi^\varepsilon(\tilde{X}_r^\beta) \exp(\nu \tilde{L}_t) \right] \right] \\
&= E_X \left[E \left[\int_0^t \exp(\nu \tilde{L}_r) \sum_{\beta \in \tilde{I}_r} \left(\prod_{\substack{\gamma \in \tilde{I}_r, \\ \gamma \neq \beta}} u^\varepsilon(s, \tilde{X}_r^\gamma) \right) \left(\frac{1}{2} \Delta u^\varepsilon(s, \tilde{X}_r^\beta) \right) ds \right] \right] \\
&\quad + \frac{1}{2} E_X \left[E \left[\int_0^t \exp(\nu \tilde{L}_r) \sum_{\substack{\beta, \gamma \in \tilde{I}_r, \\ \beta \neq \gamma}} \left(\prod_{\substack{\alpha \in \tilde{I}_r, \\ \alpha \neq \beta, \gamma}} u^\varepsilon(s, \tilde{X}_r^\alpha) \right) p_\varepsilon(y - \tilde{X}_r^\beta) p_\varepsilon(y - \tilde{X}_r^\gamma) \sigma(u(s, y)) dy ds \right] \right].
\end{aligned}$$

Now, we will see that we can apply the Fubini's theorem for the above identity and its expectation with respect to P_X . Indeed, by definition, $\exp(\nu \tilde{L}_t) \leq e^{\nu \ell}$ and

$$\begin{aligned}
& E \left[\prod_{k=1, k \neq i}^m u^\varepsilon(s, x_k) \left| \frac{1}{2} \Delta u^\varepsilon(s, x_i) \right| \right] \\
&= E \left[\frac{1}{\varepsilon} \prod_{i=1}^m u^\varepsilon(s, x_i) \right] + E \left[\int_{\mathbb{R}} \left(\prod_{k=1, k \neq i}^m u^\varepsilon(s, x_k) \right) u(s, x_i) \frac{(x_i - y)^2}{\varepsilon^2} p_\varepsilon(x_i - y) dy \right].
\end{aligned}$$

We can apply the Fubini's theorem by the following estimates: For all $\{x_i\}_{i=1}^m \subset \mathbb{R}$

$$\begin{aligned}
E \left[\prod_{i=1}^m u^\varepsilon(s, x_i) \right] &= E \left[\int_{\mathbb{R}^m} \prod_{i=1}^m u(s, y_i) p_\varepsilon(y_i - x_i) d\mathbf{y} \right] \\
&\leq \sup_{(s, x) \in [0, t] \times \mathbb{R}} E[u(s, x)^m] \left(\int_{\mathbb{R}} p_\varepsilon(y) dy \right)^m < \infty,
\end{aligned}$$

and

$$\begin{aligned} & E \left[\int_{\mathbb{R}} \left(\prod_{k=1, k \neq i}^m u^\varepsilon(s, x_k) \right) u(s, x_i) \frac{(x_i - y)^2}{\varepsilon^2} p_\varepsilon(x_i - y) dy \right] \\ & \leq \sup_{(s, x) \in [0, t] \times \mathbb{R}} E[u(s, x)^m] \int_{\mathbb{R}} \frac{(x - y)^2}{\varepsilon^2} p_\varepsilon(x - y) dy < \infty. \end{aligned}$$

Considering the compensators for the jumps in $\{\tilde{X}_t^\beta : \beta \in \tilde{I}_t\}$ at the times of coalescence, we have for $f \in C_b^2(\mathbb{R})$

$$\begin{aligned} & E_X \left[\prod_{\beta \in \tilde{I}_t} h(\tilde{X}_t^\beta) \exp(\nu \tilde{L}_t) \right] - E_X \left[\prod_{i=1}^N h(x_i) \right] \\ & = E_X \left[\int_0^t \exp(\nu \tilde{L}_s) \sum_{\beta \in \tilde{I}_s} \left(\prod_{\gamma \in \tilde{I}_s \setminus \beta} h(\tilde{X}_s^\gamma) \right) \frac{1}{2} \Delta h(\tilde{X}_s^\beta) ds \right] \\ & + E_X \left[\int_0^{t \wedge \rho_\ell} \exp(\nu \tilde{L}_s) \sum_{\substack{\beta, \gamma \in \tilde{I}_s, \\ \beta \neq \gamma}} \left(\prod_{\substack{\alpha \in \tilde{I}_s, \\ \alpha \neq \beta, \gamma}} h(\tilde{X}_s^\alpha) \right) \left(h(\tilde{X}_s^\beta) - h(\tilde{X}_s^\beta) h(\tilde{X}_s^\gamma) \right) \frac{a}{2} d\tilde{L}_s^{\beta, \gamma} \right] \\ & + E_X \left[\int_0^{t \wedge \rho_\ell} \exp(\nu \tilde{L}_s) \left(\prod_{\beta \in \tilde{I}_s} h(\tilde{X}_s^\beta) \right) \nu d\tilde{L}_s \right] \\ & = E_X \left[\int_0^t \exp(\nu \tilde{L}_s) \sum_{\beta \in \tilde{I}_s} \left(\prod_{\gamma \in \tilde{I}_s \setminus \beta} h(\tilde{X}_s^\gamma) \right) \frac{1}{2} \Delta h(\tilde{X}_s^\beta) ds \right] \\ & + E_X \left[\int_0^{t \wedge \rho_\ell} \exp(\nu \tilde{L}_s) \sum_{\substack{\beta, \gamma \in \tilde{I}_s, \\ \beta \neq \gamma}} \left(\prod_{\substack{\alpha \in \tilde{I}_s, \\ \alpha \neq \beta, \gamma}} h(\tilde{X}_s^\alpha) \right) \sigma(h(\tilde{X}_s^\beta)) d\tilde{L}_s^{\beta, \gamma} \right]. \end{aligned}$$

Since we know that $u^\varepsilon \in C_{\text{rap}}(\mathbb{R})$ and smooth function almost surely, the above identity holds with h replaced by u_r^ε . Also, the identity does hold when we take expectation with respect to P .

Set

$$g(t, s, \varepsilon) = E \left[E_X \left[\prod_{\beta \in \tilde{I}_s} u^\varepsilon(t, \tilde{X}_s^\beta) \exp(\nu \tilde{L}_s) \right] \right].$$

For any $T \geq 0$, we have that

$$\begin{aligned}
& \int_0^T (g(r, 0, \varepsilon) - g(0, r, \varepsilon)) dr \\
&= \int_0^T (g(T - r, r, \varepsilon) - g(0, r, \varepsilon)) dr - \int_0^T (g(r, T - r, \varepsilon) - g(r, 0, \varepsilon)) dr \\
&= E \left[E_X \left[\int_0^T \int_0^{T-r} \exp(\nu \tilde{L}_r) \sum_{\beta \in \tilde{I}_r} \left(\prod_{\gamma \in \tilde{I}_r \setminus \{\beta\}} u^\varepsilon(s, \tilde{X}_r^\gamma) \right) \frac{1}{2} \Delta u^\varepsilon(s, \tilde{X}_r^\beta) ds dr \right] \right] \\
&+ \frac{1}{2} E \left[E_X \left[\int_0^T \int_0^{T-r} \exp(\nu \tilde{L}_r) \sum_{\substack{\beta, \gamma \in \tilde{I}_r, \\ \beta \neq \gamma}} \left(\prod_{\substack{\alpha \in \tilde{I}_r, \\ \alpha \neq \beta, \gamma}} u^\varepsilon(s, \tilde{X}_r^\alpha) \right) \right. \right. \\
&\quad \left. \left. \times \int_{\mathbb{R}} p_\varepsilon(y - \tilde{X}_r^\beta) p_\varepsilon(y - \tilde{X}_r^\gamma) \sigma(u(s, y)) dy ds dr \right] \right] \\
&- E \left[E_X \left[\int_0^T \int_0^{T-r} \exp(\nu \tilde{L}_s) \sum_{\beta \in \tilde{I}_s} \left(\prod_{\gamma \in \tilde{I}_s \setminus \{\beta\}} u^\varepsilon(r, \tilde{X}_s^\gamma) \right) \frac{1}{2} \Delta u^\varepsilon(r, \tilde{X}_s^\beta) ds dr \right] \right] \\
&- \frac{1}{4} E \left[E_X \left[\int_0^T \int_0^{T-r} 1\{s \leq \rho_\ell\} \exp(\nu \tilde{L}_s) \sum_{\substack{\beta, \gamma \in \tilde{I}_s, \\ \beta \neq \gamma}} \left(\prod_{\substack{\alpha \in \tilde{I}_s, \\ \alpha \neq \beta, \gamma}} u^\varepsilon(r, \tilde{X}_r^\alpha) \right) \sigma(u(r, \tilde{X}_s^\beta)) d\tilde{L}_s^{\beta, \gamma} dr \right] \right],
\end{aligned}$$

and by using Fubini's theorem

$$\begin{aligned}
&= \frac{1}{2} E \left[E_X \left[\int_0^T \int_0^{T-r} \exp(\nu \tilde{L}_r) \sum_{\substack{\beta, \gamma \in \tilde{I}_r, \\ \beta \neq \gamma}} \left(\prod_{\substack{\alpha \in \tilde{I}_r, \\ \alpha \neq \beta, \gamma}} u^\varepsilon(s, \tilde{X}_r^\alpha) \right) \right. \right. \\
&\quad \left. \left. \times \int_{\mathbb{R}} p_\varepsilon(y - \tilde{X}_r^\beta) p_\varepsilon(y - \tilde{X}_r^\gamma) \sigma(u(s, y)) dy ds dr \right] \right] \tag{4.2} \\
&- \frac{1}{4} E \left[E_X \left[\int_0^T \int_0^{T-r} 1\{s \leq \rho_\ell\} \exp(\nu \tilde{L}_s) \sum_{\substack{\beta, \gamma \in \tilde{I}_s, \\ \beta \neq \gamma}} \left(\prod_{\substack{\alpha \in \tilde{I}_s, \\ \alpha \neq \beta, \gamma}} u^\varepsilon(r, \tilde{X}_r^\alpha) \right) \sigma(u(r, \tilde{X}_s^\beta)) d\tilde{L}_s^{\beta, \gamma} dr \right] \right].
\end{aligned}$$

We will show that letting $\varepsilon \rightarrow 0$, we obtain that

$$\begin{aligned}
& \int_0^T (g(r, 0, 0) - g(0, r, 0)) dr \\
&= \frac{1}{4} E \left[E_X \left[\int_0^T \int_0^{T-r} \exp(\nu \tilde{L}_r) \sum_{\substack{\beta, \gamma \in \tilde{I}_r, \\ \beta \neq \gamma}} \left(\prod_{\substack{\alpha \in \tilde{I}_r, \\ \alpha \neq \beta, \gamma}} u(s, \tilde{X}_r^\alpha) \right) \sigma(u(s, \tilde{X}_r^\beta)) ds d\tilde{L}_r^{\beta, \gamma} \right] \right] \\
& \quad (4.3) \\
& - \frac{1}{4} E \left[E_X \left[\int_0^T \int_0^{T-r} 1\{s \leq \rho_\ell\} \exp(\nu \tilde{L}_s) \sum_{\substack{\beta, \gamma \in \tilde{I}_s, \\ \beta \neq \gamma}} \left(\prod_{\substack{\alpha \in \tilde{I}_s, \\ \alpha \neq \beta, \gamma}} u(r, \tilde{X}_s^\alpha) \right) \sigma(u(r, \tilde{X}_s^\beta)) d\tilde{L}_s^{\beta, \gamma} dr \right] \right].
\end{aligned}$$

If (4.3) is true, we have that by applying change of variables and Fubini's theorem

$$\begin{aligned}
& \int_0^T (g(r, 0, 0) - g(0, r, 0)) dr \\
&= \frac{1}{4} E \left[E_X \left[\int_0^T \int_0^s 1\{t > \rho_\ell\} \exp(\nu \tilde{L}_t) \sum_{\substack{\beta, \gamma \in \tilde{I}_t, \\ \beta \neq \gamma}} \left(\prod_{\substack{\alpha \in \tilde{I}_t, \\ \alpha \neq \beta, \gamma}} u(s-t, \tilde{X}_t^\alpha) \right) \sigma(u(s-t, \tilde{X}_t^\beta)) d\tilde{L}_t^{\beta, \gamma} ds \right] \right].
\end{aligned}$$

When we verify that $g(r, 0, 0)$ is continuous at $r = T$ and $g(0, r, 0)$ is left-continuous at $r = T$, differentiating both sides of the above equation from the left at T , we have that $g(T, 0, 0) - g(0, T, 0) = \mathcal{E}$. \square

To complete the proof of Proposition 4.3, it is enough to verify the followings.

- (i) (4.3) is true.
- (ii) $g(r, 0, 0)$ is continuous at $r > 0$ and $g(0, r, 0)$ is left-continuous at $r > 0$.

Before proving them, we will prove the following lemma which tells us the upper L^2 -bound of the difference of $u(t, x) - u(t, y)$.

Lemma 4.4. *Let u be a solution of (4.1). Then, for $t \geq 0$ and $x, x' \in \mathbb{R}$*

$$E [|u(t, x) - u(t, x')|^2] \leq C(\psi) \left(\frac{|x - x'|^2}{t} + |x - x'| \right).$$

Proof. By Itô's formula,

$$u(t, x) = \int_{\mathbb{R}} u(0, y) p_t(y - x) dy + \int_0^t \int_{\mathbb{R}} p_{t-s}(y - x) \sqrt{\sigma(u(s, y))} W(ds, dy),$$

and

$$\begin{aligned}
E [|u(t, x) - u(t, x')|^2] &\leq 2 |\psi^t(x) - \psi^t(x')|^2 \\
&\quad + 2E \left[\int_0^t \int_{\mathbb{R}} (p_{t-s}(y - x) - p_{t-s}(y - x'))^2 \sigma(u(s, y)) dy ds \right].
\end{aligned}$$

We know that

$$\begin{aligned}
|\psi^t(x) - \psi^t(x')|^2 &= \left| \int_{\mathbb{R}} \psi(y)(p_t(y-x) - p_t(y-x')) dy \right|^2 \\
&\leq C(\psi) \left(\int_{\mathbb{R}} |p_t(y-x) - p_t(y-x')| dy \right)^2 \\
&\leq 4C(\psi) \frac{|x-x'|^2}{2\pi t}.
\end{aligned}$$

Also, we have that

$$\begin{aligned}
&E \left[\int_0^t \int_{\mathbb{R}} (p_{t-s}(y-x) - p_{t-s}(y-x'))^2 \sigma(u(t,y)) dy ds \right] \\
&\leq \int_0^t \int_{\mathbb{R}} (p_{t-s}(y-x) - p_{t-s}(y-x'))^2 \sup_{t,x} E [au(t,x) + bu(t,x)^2] dy ds \\
&\leq C \int_0^t \int_{\mathbb{R}} (p_{t-s}(y-x) - p_{t-s}(y-x'))^2 dy ds \leq C|x-x'|.
\end{aligned}$$

□

Lemma 4.5.

$$\lim_{\varepsilon \rightarrow 0} \int_0^T g(r, 0, \varepsilon) dr = \int_0^T g(r, 0, 0) dr,$$

and

$$\lim_{\varepsilon \rightarrow 0} \int_0^T g(0, r, \varepsilon) dr = \int_0^T g(0, r, 0) dr.$$

Proof. The latter part is trivial since we can apply the dominated convergence theorem from the fact that $|\tilde{I}_r| \leq N$, $\sup_x |\psi^\varepsilon(x)| \leq C$, and $\exp(\nu \tilde{L}_s) \leq e^{\nu \ell}$. We will look at the former part. Since $\sup_\varepsilon |g(r, 0, \varepsilon)| < \infty$, it is enough to show that $g(r, 0, \varepsilon) \rightarrow g(r, 0, 0)$ for each $r > 0$.

$$\begin{aligned}
g(r, 0, \varepsilon) - g(r, 0, 0) &= E \left[\prod_{i=1}^N u^\varepsilon(r, x_i) \right] - E \left[\prod_{i=1}^N u(r, x_i) \right] \\
&= E \left[\sum_{i=1}^N \left(\prod_{k=1}^{i-1} u^\varepsilon(r, x_k) \right) (u^\varepsilon(r, x_i) - u(r, x_i)) \left(\prod_{j=i+1}^N u(r, x_j) \right) \right],
\end{aligned}$$

where we have used that $\prod_{i=1}^N (a_i + b_i) - \prod_{i=1}^N a_i = \sum_{k=1}^N \left(\prod_{i=1}^{k-1} (a_i + b_i) \right) b_k \left(\prod_{j=k+1}^N a_j \right)$.

Also, we have that for each $1 \leq k \leq N$

$$\begin{aligned}
& \left| E \left[\left(\prod_{i=1}^{k-1} u^\varepsilon(r, x_i) \right) (u^\varepsilon(r, x_k) - u(r, x_k)) \left(\prod_{j=k+1}^N u(r, x_j) \right) \right] \right| \\
& \leq \int_{\mathbb{R}^k} E \left[\left(\prod_{i=1}^{k-1} u(r, y_i + x_i) p_\varepsilon(y_i) \right) |u(r, y_k + x_k) - u(r, x_k)| p_\varepsilon(y_k) \left(\prod_{j=k+1}^N u(r, x_j) \right) d\mathbf{y} \right] \\
& \leq \int_{\mathbb{R}^k} \prod_{j=1}^k p_\varepsilon(y_j) d\mathbf{y} \sup_{t,x} E \left[u(s, x)^{2(N-1)} \right]^{\frac{1}{2}} E \left[|u(r, x_k + y_k) - u(r, x_k)|^2 \right]^{\frac{1}{2}} \\
& \leq C(\psi) \sqrt{\varepsilon}.
\end{aligned}$$

Thus, we have proved $g(r, 0, \varepsilon) \rightarrow g(r, 0, 0)$ for any $r > 0$ and the statement follows. \square

Lemma 4.6.

$$\begin{aligned}
& E \left[E_X \left[\int_0^T \int_0^{T-r} 1\{s \leq \rho_\ell\} \exp(\nu \tilde{L}_s) \sum_{\substack{\beta, \gamma \in \tilde{I}_s, \\ \beta \neq \gamma}} \left(\prod_{\substack{\alpha \in \tilde{I}_s, \\ \alpha \neq \beta, \gamma}} u^\varepsilon(r, \tilde{X}_r^\alpha) \right) \sigma(u(r, \tilde{X}_s^\beta)) d\tilde{L}_s^{\beta, \gamma} dr \right] \right] \\
& \rightarrow E \left[E_X \left[\int_0^T \int_0^{T-r} 1\{s \leq \rho_\ell\} \exp(\nu \tilde{L}_s) \sum_{\substack{\beta, \gamma \in \tilde{I}_s, \\ \beta \neq \gamma}} \left(\prod_{\substack{\alpha \in \tilde{I}_s, \\ \alpha \neq \beta, \gamma}} u(r, \tilde{X}_s^\alpha) \right) \sigma(u(r, \tilde{X}_s^\beta)) d\tilde{L}_s^{\beta, \gamma} dr \right] \right].
\end{aligned}$$

Proof. By Fubini's theorem,

$$\begin{aligned}
& E \left[E_X \left[\int_0^T \int_0^{T-r} 1\{s \leq \rho_\ell\} \exp(\nu \tilde{L}_s) \sum_{\substack{\beta, \gamma \in \tilde{I}_s, \\ \beta \neq \gamma}} \left(\prod_{\substack{\alpha \in \tilde{I}_s, \\ \alpha \neq \beta, \gamma}} u^\varepsilon(r, \tilde{X}_r^\alpha) \right) \sigma(u(r, \tilde{X}_s^\beta)) d\tilde{L}_s^{\beta, \gamma} dr \right] \right] \\
& = E_X \left[\int_0^T \int_0^{T-r} 1\{s \leq \rho_\ell\} \exp(\nu \tilde{L}_s) \sum_{\substack{\beta, \gamma \in \tilde{I}_s, \\ \beta \neq \gamma}} E \left[\left(\prod_{\substack{\alpha \in \tilde{I}_s, \\ \alpha \neq \beta, \gamma}} u^\varepsilon(r, \tilde{X}_r^\alpha) \right) \sigma(u(r, \tilde{X}_s^\beta)) \right] d\tilde{L}_s^{\beta, \gamma} dr \right].
\end{aligned}$$

It is easy to see that the integrand is finite almost everywhere P_X -almost surely. Thus, If we find that

$$\sum_{\substack{\beta, \gamma \in \tilde{I}_s, \\ \beta \neq \gamma}} E \left[\left(\prod_{\substack{\alpha \in \tilde{I}_s, \\ \alpha \neq \beta, \gamma}} u^\varepsilon(r, \tilde{X}_r^\alpha) \right) \sigma(u(r, \tilde{X}_s^\beta)) \right] \rightarrow \sum_{\substack{\beta, \gamma \in \tilde{I}_s, \\ \beta \neq \gamma}} E \left[\left(\prod_{\substack{\alpha \in \tilde{I}_s, \\ \alpha \neq \beta, \gamma}} u(r, \tilde{X}_r^\alpha) \right) \sigma(u(r, \tilde{X}_s^\beta)) \right],$$

almost everywhere P_X -almost surely, the statement follows. However, we have already proved it in the proof of Lemma 4.5. \square

To prove (4.3), we will use the following lemma (see [1, Lemma 2]).

Lemma 4.7. Consider two independent Brownian motions B_t^1 and B_t^2 adapted to a filtration \mathcal{G}_t . Let $f : \mathbb{R} \rightarrow [0, \infty)$ be a bounded continuous function, $X_t = B_t^1 - B_t^2$, and Y_t be a (\mathcal{G}_t) -predictable process satisfying $E[\int_0^t |f(s)| ds] < \infty$. Then, for $t \geq 0$

$$2 \int_0^t f(X_s) Y_s ds = \int \int_0^t f(z) Y_s dL_{s,z}^X dz \quad a.s.,$$

where $L_{s,z}^X$ is the local time of X at z until time s .

Lemma 4.8. Letting $\varepsilon \rightarrow 0$, then

$$(4.2) \rightarrow \frac{1}{4} E_X \left[E \left[\int_0^T \int_0^{T-r} \exp(\nu \tilde{L}_r) \sum_{\substack{\beta, \gamma \in \tilde{I}_r, \\ \beta \neq \gamma}} \left(\prod_{\substack{\alpha \in \tilde{I}_r, \\ \alpha \neq \beta, \gamma}} u(s, \tilde{X}_r^\alpha) \right) \sigma(u(s, \tilde{X}_r^\beta)) ds d\tilde{L}_r^{\beta, \gamma} \right] \right].$$

Proof. We set

$$Y_{r,\varepsilon}^{\beta, \gamma}(y) = 1\{\beta, \gamma \in \tilde{I}_r\} \int_0^{T-r} \exp(\nu \tilde{L}_r) E \left[\left(\prod_{\substack{\alpha \in \tilde{I}_r, \\ \alpha \neq \beta, \gamma}} u^\varepsilon(s, \tilde{X}_r^\alpha) \right) \sigma(u(s, y + \tilde{X}_r^\beta)) ds \right].$$

Then, we can write

$$\begin{aligned} & E \left[\int_0^T \int_0^{T-r} \exp(\nu \tilde{L}_r) \sum_{\substack{\beta, \gamma \in \tilde{I}_r, \\ \beta \neq \gamma}} \left(\prod_{\substack{\alpha \in \tilde{I}_r, \\ \alpha \neq \beta, \gamma}} u^\varepsilon(s, \tilde{X}_r^\alpha) \right) \times \int_{\mathbb{R}} p_\varepsilon(y - \tilde{X}_r^\beta) p_\varepsilon(y - \tilde{X}_r^\gamma) \sigma(u(s, y)) ds dr \right] \\ &= \sum_{\beta, \gamma \in 2^{I_0}, \beta \neq \gamma} \int_0^T \int p_\varepsilon(y + \tilde{X}_r^\beta - \tilde{X}_r^\gamma) p_\varepsilon(y) Y_{r,\varepsilon}^{\beta, \gamma}(y) dr dy. \end{aligned} \quad (4.4)$$

Let \mathcal{G}_t be the filtration generated by $(\tilde{X}_r^\beta, \tilde{I}_r)$ up to time t . Then, $Y_{r,\varepsilon}^{\beta, \gamma}(y)$ is \mathcal{G}_r -adapted and left continuous. Since $\sup_y Y_{r,\varepsilon}^{\beta, \gamma}(y)$ is bounded above by CT , we can apply Lemma 4.7, and for $y \in \mathbb{R}$,

$$\int_0^T p_\varepsilon(y + \tilde{X}_r^\beta - \tilde{X}_r^\gamma) Y_{r,\varepsilon}^{\beta, \gamma}(y) dr = \frac{1}{2} \int \int_0^T p_\varepsilon(y + z) Y_{r,\varepsilon}^{\beta, \gamma}(y) d\tilde{L}_{r,z}^{\beta, \gamma} dz.$$

Therefore, the integrand in (4.2) is rewritten as

$$\frac{1}{2} \sum_{\beta, \gamma \in 2^{I_0}, \beta \neq \gamma} \int \int \int_0^T p_\varepsilon(y + z) p_\varepsilon(y) Y_{r,\varepsilon}^{\beta, \gamma}(y) d\tilde{L}_{r,z}^{\beta, \gamma} dz dy.$$

By the same argument as the proof of Lemma 4.5, we have that P_X -almost surely

$$\begin{aligned} |Y_{r,\varepsilon}^{\beta, \gamma}(y) - Y_{r,\varepsilon}^{\beta, \gamma}(0)| &\leq 1\{\beta, \gamma \in \tilde{I}_r\} C \int_0^{T-r} \exp(\nu \ell) \left(\frac{|y|}{\sqrt{s}} + \sqrt{|y|} \right) ds \\ &\leq 1\{\beta, \gamma \in \tilde{I}_r\} C (\sqrt{T-r}|y| + (T-r)\sqrt{|y|}). \end{aligned}$$

Thus, for fixed $\beta, \gamma \in 2^{I_0}$, the path $y \rightarrow Y_{r,\varepsilon}^{\beta,\gamma}(y)$ is continuous at 0 uniformly in $r \leq T$. Also, $\sup_{r < T, y} |Y_{r,\varepsilon}^{\beta,\gamma}(y) - Y_{r,0}^{\beta,\gamma}(y)| \leq C\sqrt{\varepsilon}$ as $\varepsilon \rightarrow 0$.

Hence, it tells us that

$$\begin{aligned}
& \left| \int \int \int_0^T p_\varepsilon(y+z) p_\varepsilon(y) Y_{r,\varepsilon}^{\beta,\gamma}(y) d\tilde{L}_{r,z}^{\beta,\gamma} dz dy - \int \int \int_0^T p_\varepsilon(y+z) p_\varepsilon(y) Y_{r,0}^{\beta,\gamma}(0) d\tilde{L}_{r,z}^{\beta,\gamma} dz dy \right| \\
& \leq \int \int \int_0^T |p_\varepsilon(y+z) p_\varepsilon(y)| C(\sqrt{T}|y| + T\sqrt{|y|}) d\tilde{L}_{r,z}^{\beta,\gamma} dz dy + \int \int \int_0^T C\sqrt{\varepsilon} p_\varepsilon(y+z) p_\varepsilon(y) d\tilde{L}_{r,z}^{\beta,\gamma} dz dy \\
& = \int \int_0^T |p_\varepsilon(y + \tilde{X}_r^\beta - \tilde{X}_r^\gamma) p_\varepsilon(y)| C(\sqrt{T}|y| + T\sqrt{y} + \varepsilon) dr dy \\
& \leq \int_0^T C p_\varepsilon(\tilde{X}_r^\beta - \tilde{X}_r^\gamma) \left(\varepsilon + \sqrt{\varepsilon} + \varepsilon^{\frac{1}{4}} + \frac{|\tilde{X}_r^\beta - \tilde{X}_r^\gamma|^{\frac{3}{2}} + |\tilde{X}_r^\beta - \tilde{X}_r^\gamma|^2}{\sqrt{\varepsilon}} \right) dr \\
& \rightarrow 0, \text{ as } \varepsilon \rightarrow 0, \text{ for all } \tilde{X}_r^\beta - \tilde{X}_r^\gamma.
\end{aligned}$$

Also, we have that

$$\begin{aligned}
& E_X \left[\left| \int \int \int_0^T p_\varepsilon(y+z) p_\varepsilon(y) Y_{r,\varepsilon}^{\beta,\gamma}(y) d\tilde{L}_{r,z}^{\beta,\gamma} dz dy - \int \int \int_0^T p_\varepsilon(y+z) p_\varepsilon(y) Y_{r,0}^{\beta,\gamma}(0) d\tilde{L}_{r,z}^{\beta,\gamma} dz dy \right| \right] \\
& \leq CT(\varepsilon + \sqrt{\varepsilon} + \varepsilon^{\frac{1}{4}}) \rightarrow 0.
\end{aligned}$$

It is clear that

$$\begin{aligned}
& \int \int \int_0^T p_\varepsilon(x+y) p_\varepsilon(y) Y_{r,0}^{\beta,\gamma}(0) d\tilde{L}_{r,z}^{\beta,\gamma} dz dy \\
& = \int \int_0^T \frac{1}{2\sqrt{\pi\varepsilon}} \exp\left(-\frac{z^2}{4\varepsilon}\right) Y_{r,0}^{\beta,\gamma}(0) d\tilde{L}_{r,z}^{\beta,\gamma} dr dz \\
& \rightarrow \int_0^T Y_{r,0}^{\beta,\gamma}(0) d\tilde{L}_r^{\beta,\gamma}, \text{ as } \varepsilon \rightarrow 0,
\end{aligned}$$

P_X -almost surely by the continuity of $Y_{r,0}^{\beta,\gamma}(0)$ and the continuity of the local time. Also, since we have $E_X \left[\sup_z \tilde{L}_{T,z}^{\beta,\gamma} \right] < \infty$ for each $\beta, \gamma \in 2^{I_0}$, we can apply the dominated convergence theorem, and the statement follows. \square

Lemma 4.9. $g(r, 0, 0)$ is continuous at $r > 0$ and $g(0, r, 0)$ is left continuous at $r > 0$.

Proof. $g(r, 0, 0)$ is continuous by the continuity of $u(t, x)$ in $t \geq 0$ and by dominated convergence theorem. $g(0, r, 0)$ is left continuous by left continuity of \tilde{X}_s and the continuity and the boundedness of \tilde{L}_s . \square

Thus, we have prove that Proposition 4.3.

We now turn to estimate \mathcal{E} .

Lemma 4.10. ([1, Lemma 3]) Suppose that $p, q \in \mathbb{R}_+$ satisfy that

$$p + \frac{a}{2}e^{-q} < \frac{a}{2}$$

Then, for all $T \geq 0$,

$$E_X \left[\sup_{0 \leq t \leq T} \exp(pL_t + q|I_t|) \right] < \infty,$$

where $|I_t|$ is the number of elements of I_t .

The following proof is the same as the one in [1]. However, we give it here since it gives us a useful estimate which we will use later.

Proof. Considering the jumps of $|I_t|$ we see the compensator of the pure jump process $\exp(q|I_t|)$ is given by

$$\int_0^t \frac{a}{2} (\exp(q(|I_s| - 1)) - \exp(q|I_s|)) dL_s = \left(\frac{a}{2} e^{-q} - \frac{a}{2} \right) \int_0^t \exp(q|I_s|) ds.$$

Set $Z_t = \exp(pL_t + q|I_t|)$. By Itô's formula, we have that

$$dZ_t = \left(p + \frac{a}{2} (e^{-q} - 1) \right) Z_t dL_t + dM_t,$$

where M_t is a local martingale. By the assumption, for all $\theta > 1$, $Z_t^\theta = \exp(\theta pL_t + \theta q|I_t|)$ is a non-negative supermartingale. This implies, by Doob's optional stopping time argument, that $P_X(\sup_{t \geq 0} Z_t^\theta \geq c) \leq \frac{\exp(\theta q|I_0|)}{c}$ and the conclusion of the lemma follows. \square

By the above argument, for $1 < \theta < \theta'$

$$P_X \left(\sup_{t \leq T} Z_t^\theta > c \right) \leq \frac{\exp(\theta' qN)}{c^{\frac{\theta'}{\theta}}}$$

and

$$E_X \left[\sup_{t \leq T} Z_t^\theta \right] \leq C \exp(\theta' qN). \quad (4.5)$$

Proof of Theorem 4.1. Letting $\ell \rightarrow \infty$, $\{\tilde{X}_s^\beta : \beta \in \tilde{I}_s, s \in [0, T]\}$ converges to $\{X_s^\beta : \beta \in I_s, s \in [0, T]\}$. By the boundedness of ϕ and $|I_s| \leq N$, and by Lemma 4.10, we have that

$$E_X \left[\prod_{\beta \in \tilde{I}_T} \psi(\tilde{X}_T^\beta) \exp(\nu \tilde{L}_T) \right] \rightarrow E_X \left[\prod_{\beta \in I_T} \psi(X_T^\beta) \exp(\nu L_T) \right], \text{ as } \ell \rightarrow \infty.$$

Thus, it is enough to show that $\mathcal{E} \rightarrow 0$ as $\ell \rightarrow \infty$.

Also, by the above argument, we have

$$\begin{aligned} \mathcal{E} &\leq C E_X \left[\int_{\rho_\ell}^T \sup_{t \leq T} \exp(\nu \tilde{L}_t) d\tilde{L}_t \right] \\ &\leq C E_X [\exp(\nu L_T) (L_T - L_{\rho_\ell})]. \end{aligned}$$

When $\ell \rightarrow \infty$, $L_T - L_{\rho_\ell} \rightarrow 0$. Also, by (4.5), we can apply the dominated convergence theorem so that $\mathcal{E} \rightarrow 0$. \square

Thus, we have proved the existence of the moment duality for solutions of (4.1). In the end, we will prove Theorem 2.2.

Proof of Theorem 2.2. Let $C_c^+(\mathbb{R})$ be the set of the non-negative functions with compact support. Let $\phi \in C_c^+(\mathbb{R})$. Then, by Fubini's theorem we have

$$\begin{aligned} E[\langle u_t, \phi \rangle^N] &= E \left[\int_{\mathbb{R}^N} \prod_{i=1}^N u_t(x_i) \phi(x_i) d\mathbf{x} \right] \\ &= \int_{\mathbb{R}^N} E_X \left[\prod_{\beta \in I_t} \psi(X_t^\beta) \exp(\nu L_t) \right] \prod_{i=1}^N \phi(x_i) d\mathbf{x}. \end{aligned}$$

Since we assumed that ϕ has a compact support and ψ is bounded, the last term is bounded above by

$$C(\psi)^N |K|^N \sup_{\mathbf{x} \in K^N} E_X [\exp(\nu L_t)], \quad (4.6)$$

where K is the compact set such that $\text{supp}(\phi) \subset K$. By (4.5), we have that

$$(4.6) \leq CC(\psi)^N |K|^N \exp(\theta' q N).$$

Thus, we obtain that

$$\limsup_{N \rightarrow \infty} \frac{(E[\langle u_t, \phi \rangle^N])^{\frac{1}{N}}}{N} = 0,$$

and the Carleman's condition is satisfied.

Thus, for each $\phi \in C_c^+(\mathbb{R})$, $E[\exp(-\langle u_t, \phi \rangle)]$ is independent of the choice of the solutions of (4.1). The closure of $C_c^+(\mathbb{R})$ under the bounded pointwise convergence is the set of the bounded Borel functions, $b\mathcal{E}^+$. So, for $u(t, x)$ and $u'(t, x)$, solutions of (4.1), for $\phi \in b\mathcal{E}^+$, we have that

$$E[\exp(-\langle u_t, \phi \rangle)] = E[\exp(-\langle u'_t, \phi \rangle)]$$

and the weak uniqueness follows. \square

5 Proof of some facts

This section is devoted to the proof of some lemmas used in section 3.

Lemma 5.1. *For any $\beta > 0$ and $K > 0$, we have that*

$$\sup_N E_{Y^1 Y^2} \left[\left(1 + \frac{\beta^2}{N^{\frac{1}{2}}} \right)^{\#\{1 \leq i \leq \lfloor KN \rfloor : Y_i^1 = Y_i^2\}} \right] < \infty,$$

where Y_n^1, Y_n^2 are independent simple random walks on \mathbb{Z} . Also,

$$\begin{aligned} E_{Y^1 Y^2} \left[\left(1 + \frac{\beta^2}{N^{\frac{1}{2}}} \right)^{\#\{1 \leq i \leq \lfloor KN \rfloor : Y_i^1 = Y_i^2\}} : Y_{\lfloor KN \rfloor}^1 = x, Y_{\lfloor KN \rfloor}^2 = y \right] \\ \leq \frac{C}{K^{\frac{1}{2}} N^{\frac{1}{2}}} \left(P_{Y^1} (Y_{\lfloor KN \rfloor}^1 = x) \wedge P_{Y^1} (Y_{\lfloor KN \rfloor}^1 = y) \right). \end{aligned}$$

Proof. First, we remark that

$$\begin{aligned}
E_{Y^1 Y^2} \left[\left(1 + \frac{\beta^2}{N^{\frac{1}{2}}} \right)^{\#\{1 \leq i \leq \lfloor KN \rfloor : Y_i^1 = Y_i^2\}} \right] &= E_{Y^1 Y^2} \left[\prod_{k=1}^{\lfloor KN \rfloor} \left(1 + \frac{\beta^2}{N^{\frac{1}{2}}} \mathbf{1}_{\{Y_k^1 = Y_k^2\}} \right) \right] \\
&= \sum_{k=0}^{\infty} \frac{\beta^{2k}}{N^{\frac{k}{2}}} \sum_{\mathbf{i} \in D^k(\lfloor KN \rfloor)} \sum_{\mathbf{x} \in \mathbb{Z}^k} P_{Y^1 Y^2} (Y_{i_j}^1 = Y_{i_j}^2 = x_j, \text{ for } 1 \leq j \leq k) \\
&= \sum_{k=0}^{\infty} \frac{\beta^{2k}}{N^{\frac{k}{2}}} \sum_{\mathbf{i} \in D^k(\lfloor KN \rfloor)} \sum_{\mathbf{x} \in \mathbb{Z}^k} P_Y (Y_{i_j} = x_j, \text{ for } 1 \leq j \leq k)^2, \tag{5.1}
\end{aligned}$$

where $D^k(\lfloor KN \rfloor)$ is the set defined by

$$D^k(n) = \{\mathbf{i} = (i_j)_{j=1}^k \in \mathbb{N}^k : 1 \leq i_1 < \dots < i_k \leq n\},$$

and the summation for $k > \lfloor KN \rfloor$ is equal to 0. By the local limit theorem

$$\begin{aligned}
&\sum_{\mathbf{i} \in D^k(\lfloor KN \rfloor)} \sum_{\mathbf{x} \in \mathbb{Z}^k} P_Y (Y_{i_j} = x_j, \text{ for } 1 \leq j \leq k)^2 \\
&\leq C^k \sum_{\mathbf{i} \in D^k(\lfloor KN \rfloor)} \sum_{\mathbf{x} \in \mathbb{Z}^k} \prod_{j=1}^k \frac{P_Y (Y_{i_j - i_{j-1}} = x_j - x_{j-1})}{\sqrt{i_j - i_{j-1}}} \\
&\leq C^k \sum_{\mathbf{i} \in D^k(\lfloor KN \rfloor)} \prod_{j=1}^k \frac{1}{\sqrt{i_j - i_{j-1}}}.
\end{aligned}$$

Thus, we have that

$$(5.1) \leq \sum_{k=0}^{\infty} \frac{\beta^{2k} C^k}{N^k} \sum_{\mathbf{i} \in D^k(\lfloor KN \rfloor)} \prod_{j=1}^k \frac{1}{\sqrt{\frac{i_j}{N} - \frac{i_{j-1}}{N}}}. \tag{5.2}$$

Since $\frac{1}{\sqrt{t-s}}$ is decreasing in $t \in (s, \infty)$, it follows that

$$\frac{1}{N^k} \prod_{j=1}^k \frac{1}{\sqrt{\frac{i_j}{N} - \frac{i_{j-1}}{N}}} \leq \prod_{i=1}^k \int_{\frac{i_{j-1}}{N}}^{\frac{i_j}{N}} \frac{dt_j}{\sqrt{t_j - \frac{i_{j-1}}{N}}},$$

and

$$\begin{aligned}
(5.2) &\leq \sum_{k=0}^{\infty} \beta^{2k} C^k \sum_{\mathbf{i} \in D^k(\lfloor KN \rfloor)} \int_{\frac{i_{k-1}}{N}}^{\frac{i_k}{N}} \dots \int_0^{\frac{i_1}{N}} \prod_{j=1}^k \left(\frac{1}{\sqrt{t_j - \frac{i_{j-1}}{N}}} \right) d\mathbf{t} \\
&\leq \sum_{k=0}^{\infty} \beta^{2k} C^k \int_{0 < t_1 < \dots < t_k < K} \prod_{j=1}^k \frac{1}{\sqrt{t_j - t_{j-1}}} d\mathbf{t} \\
&= \sum_{k=0}^{\infty} \frac{\beta^{2k} C^k (\pi K)^{\frac{k}{2}}}{\Gamma(\frac{k}{2} + 1)}.
\end{aligned}$$

Since $\Gamma\left(\frac{k}{2} + 1\right)$ is increase faster than a^k for any $a > 1$, the summation is finite for any β .

Also, the similar argument does hold so that

$$\begin{aligned}
& E_{Y^1 Y^2} \left[\left(1 + \frac{\beta^2}{N^{\frac{1}{2}}} \right)^{\#\{1 \leq i \leq \lfloor KN \rfloor : Y_i^1 = Y_i^2\}} : Y_{\lfloor KN \rfloor}^1 = x, Y_{\lfloor KN \rfloor}^2 = y \right] \\
&= \sum_{k=1}^{\infty} \frac{\beta^{2(k-1)}}{N^{\frac{k-1}{2}}} \sum_{\mathbf{i} \in D^{k-1}(\lfloor KN \rfloor - 1)} \sum_{\mathbf{x} \in \mathbb{Z}^{k-1}} \left(P_Y(Y_{i_j} = x_j, \text{ for } 1 \leq j \leq k-1, Y_{\lfloor KN \rfloor} = x) \right. \\
&\quad \left. \times P_Y(Y_{i_j} = x_j, \text{ for } 1 \leq j \leq k-1, Y_{\lfloor KN \rfloor} = y) \right) \\
&\quad + \sum_{k=1}^{\infty} \frac{\beta^{2k}}{N^{\frac{k}{2}}} \sum_{\mathbf{i} \in D^{k-1}(\lfloor KN \rfloor - 1)} \sum_{\mathbf{x} \in \mathbb{Z}^{k-1}} \left(P_Y(Y_{i_j} = x_j, \text{ for } 1 \leq j \leq k-1, Y_{\lfloor KN \rfloor} = x) \right. \\
&\quad \left. \times P_Y(Y_{i_j} = x_j, \text{ for } 1 \leq j \leq k-1, Y_{\lfloor KN \rfloor} = y) \right) \\
&\leq \sum_{k=1}^{\infty} 2C^k \frac{\beta^{2(k-1)}}{N^{\frac{k-1}{2}}} \sum_{\mathbf{i} \in D^{k-1}(\lfloor KN \rfloor - 1)} \left(\prod_{j=1}^{k-1} \frac{1}{\sqrt{i_j - i_{j-1}}} \right) \frac{P_Y(Y_{\lfloor KN \rfloor}^1 = x) \wedge P_Y(Y_{\lfloor KN \rfloor}^1 = y)}{\sqrt{\lfloor KN \rfloor - i_{k-1}}} \\
&\leq \sum_{k=1}^{\infty} \frac{C^k \beta^{2(k-1)}}{N^{\frac{1}{2}}} \frac{P_Y(Y_{\lfloor KN \rfloor}^1 = x) \wedge P_Y(Y_{\lfloor KN \rfloor}^1 = y)}{N^{k-1}} \sum_{\mathbf{i} \in D^{k-1}(\lfloor KN \rfloor - 1)} \prod_{j=1}^{k-1} \frac{1}{\sqrt{\frac{i_j}{N} - \frac{i_{j-1}}{N}}} \frac{1}{\sqrt{K - \frac{i_{k-1}}{N}}}.
\end{aligned} \tag{5.3}$$

By the integration by parts, we have that

$$\begin{aligned}
\int_{\frac{i_{k-2}}{N}}^{\frac{i_{k-1}}{N}} \frac{1}{\sqrt{t_{k-1} - \frac{i_{k-2}}{N}} \sqrt{K - t_{k-1}}} dt_{k-1} &= \left[2 \frac{\sqrt{t_{k-1} - \frac{i_{k-2}}{N}}}{\sqrt{K - t_{k-1}}} \right]_{\frac{i_{k-2}}{N}}^{\frac{i_{k-1}}{N}} + \text{positive term} \\
&\geq 2 \frac{\sqrt{\frac{i_{k-1}}{N} - \frac{i_{k-2}}{N}}}{\sqrt{K - \frac{i_{k-1}}{N}}} \\
&\geq \frac{2}{N} \frac{1}{\sqrt{\frac{i_{k-1}}{N} - \frac{i_{k-2}}{N}} \sqrt{K - \frac{i_{k-1}}{N}}}.
\end{aligned}$$

Also, we know that

$$\begin{aligned}
& \sum_{\mathbf{i} \in D^{k-1}(\lfloor KN \rfloor - 1)} \left(\prod_{j=1}^{k-2} \int_{\frac{i_{j-1}}{N}}^{\frac{i_j}{N}} \frac{1}{\sqrt{t_j - \frac{i_{j-1}}{N}}} dt_j \right) \left(\int_{\frac{i_{k-2}}{N}}^{\frac{i_{k-1}}{N}} \frac{1}{\sqrt{t_{k-1} - \frac{i_{k-2}}{N}} \sqrt{K - t_{k-1}}} dt_{k-1} \right) \\
&\leq \int_{0 < t_1 < \dots < t_{k-1} < K} \prod_{j=1}^{k-1} \left(\frac{1}{\sqrt{t_j - t_{j-1}}} \right) \frac{1}{\sqrt{K - t_{k-1}}} dt \\
&\leq \frac{\pi^{\frac{k}{2}} K^{\frac{k-1}{2}}}{K^{\frac{1}{2}} \Gamma\left(\frac{k-1}{2}\right)}.
\end{aligned}$$

Thus, we have that

$$\begin{aligned}
& (5.3) \\
&\leq \frac{P_Y(Y_{\lfloor KN \rfloor}^1 = x) \wedge P_Y(Y_{\lfloor KN \rfloor}^1 = y)}{(KN)^{\frac{1}{2}}} \sum_{k=1}^{\infty} \frac{C^k \beta^{2(k-1)} K^{\frac{k-1}{2}}}{\Gamma\left(\frac{k-1}{2}\right)}.
\end{aligned}$$

Since the summation is finite for any $\beta \in \mathbb{R}$, the statement holds. \square

The next lemma gives us an upper bound of p -th moment of B_n for branching random walks in random environment.

Lemma 5.2. *If $E[m_{n,x}^{(p)}] = K < \infty$ for $p \in \mathbb{N}$ and $E[m_{n,x}^{(1)}] = 1$, then*

$$E[B_n^p] \leq C(p, K)n^{p-1}E_{Y^1 \dots Y^p} \left[E \left[\left(m_{0,0}^{(1)} \right)^p \right]^{\#\{1 \leq i \leq n: Y_i^a = Y_i^b, a \neq b \in \{1, \dots, p\}\}} \right]$$

and

$$E \left[\prod_{i=1}^p B_{n,x_i} \right] \leq C(p, K)n^{p-1}E_{Y^1 \dots Y^p} \left[E \left[\left(m_{0,0}^{(1)} \right)^p \right]^{\#\{1 \leq i \leq n: Y_i^a = Y_i^b, a \neq b \in \{1, \dots, p\}\}} : Y_n^i = x_i \text{ for } 1 \leq i \leq p \right].$$

Before starting a proof, we give another representation of B_n . Let $\{V_{n,x}^{\mathbf{x}} : \mathbf{x} \in \mathcal{T}, (n, x) \in \mathbb{N} \times \mathbb{Z}^d\}$ be \mathbb{N} -valued random variables with $P(V_{n,x}^{\mathbf{x}} = k | \omega) = q_{n,x}(k)$. Let $\{X_{n,x}^{\mathbf{x}} : \mathbf{x} \in \mathcal{T}, (n, x) \in \mathbb{N} \times \mathbb{Z}^d\}$ be i.i.d. random variables with $P(X_{n,x}^{\mathbf{x}} = e) = \frac{1}{2d}$ for $e = \pm e_j, j = 1, \dots, d$ where e_j are unit vector on \mathbb{Z}^d . $V_{n,x}^{\mathbf{x}}$ denotes the number of offsprings of \mathbf{x} if \mathbf{x} locates at x at time n and $X_{n,x}^{\mathbf{x}}$ denotes the step of \mathbf{x} if it locates at x at time n .

We consider the event $\{\text{particle } \mathbf{y} \text{ exists and locates at site } y \text{ at time } |\mathbf{x}| = n\}$ and its indicator function

$$B_{n,y}^{\mathbf{y}} = \mathbf{1} \{\text{particle } \mathbf{y} \text{ exists and locates at site } y \text{ at time } |\mathbf{x}| = n\}.$$

Then, it is clear that

$$\begin{aligned} B_{0,x}^{\mathbf{x}} &= \delta_{x,\mathbf{x}} = \begin{cases} 1 & \text{if } x = 0 \text{ and } \mathbf{x} = \mathbf{1}, \\ 0 & \text{otherwise,} \end{cases} \\ B_{n,y}^{\mathbf{y}} &= \sum_{x,\mathbf{x}} B_{n-1,x}^{\mathbf{x}} \mathbf{1} \{X_{n-1,x}^{\mathbf{x}} = y - x, V_{n-1,x}^{\mathbf{y}} \geq y/\mathbf{x} \geq 1\} \\ &= \sum_{0 \rightarrow y} \sum_{\mathbf{1} \rightarrow \mathbf{y}} \prod_{i=0}^{n-1} \mathbf{1} \{X_{i,y_i}^{\mathbf{y}_i} = y_{i+1} - y_i, V_{i,y_i}^{\mathbf{y}_i} \geq y_{i+1}/y_i \geq 1\}, \end{aligned}$$

and

$$B_{n,y} = \sum_{\mathbf{y}} \sum_{0 \rightarrow y} \sum_{\mathbf{1} \rightarrow \mathbf{y}} \prod_{i=0}^{n-1} \mathbf{1} \{X_{i,y_i}^{\mathbf{y}_i} = y_{i+1} - y_i, V_{i,y_i}^{\mathbf{y}_i} \geq y_{i+1}/y_i \geq 1\}.$$

We introduce new Markov chain $\mathbf{Y} = (Y, \mathbb{Y})$ on $\mathbb{Z}^d \times \mathcal{T}$ which are determined by

$$Y_0 = 0, \mathbb{Y}_0 = \mathbf{1} \in T_0.$$

$$P_{Y\mathbb{Y}} \left(\begin{matrix} Y_{n+1}=y, \\ \mathbb{Y}_{n+1}=\mathbf{y} \end{matrix} \middle| \begin{matrix} Y_n=x, \\ \mathbb{Y}_n=\mathbf{x} \end{matrix} \right) = \begin{cases} \frac{1}{2d} \sum_{k \geq y/\mathbf{x}} q(k) & \text{if } |y-x|=1, y/\mathbf{x} < \infty, \\ 0 & \text{otherwise,} \end{cases}$$

where $q(k) = E[q_{n,x}(k)]$. Let $A_{n,x,y}^{\mathbf{x},\mathbf{y}} = \mathbf{1}\{X_{n,x}^{\mathbf{x}} = y - x, V^{\mathbf{x}} \geq \mathbf{y}/\mathbf{x}\}$. Then, we have the following representation of $B_{n,y}$ [20]:

$$B_{n,y} = E_{Y\mathbb{Y}} \left[\prod_{i=0}^{n-1} \frac{A_{i,Y_i,Y_{i+1}}^{\mathbb{Y}_i,\mathbb{Y}_{i+1}}}{E \left[A_{i,Y_i,Y_{i+1}}^{\mathbb{Y}_i,\mathbb{Y}_{i+1}} \right]} : Y_n = y \right],$$

and also

$$E \left[\prod_{i=1}^p B_{n,x_i} \right] = E_{\mathbf{Y}^1 \dots \mathbf{Y}^p} \left[\prod_{i=0}^{n-1} E \left[\frac{\prod_{j=1}^p A_{i,Y_i^j,Y_{i+1}^j}^{\mathbb{Y}_i^j,\mathbb{Y}_{i+1}^j}}{\prod_{j=1}^p E \left[A_{i,Y_i^j,Y_{i+1}^j}^{\mathbb{Y}_i^j,\mathbb{Y}_{i+1}^j} \right]} : Y_n^i = x_i \text{ for } 1 \leq i \leq p \right] \right]$$

$$E[B_n^p] = E_{\mathbf{Y}^1 \dots \mathbf{Y}^p} \left[\prod_{i=0}^{n-1} E \left[\frac{\prod_{j=1}^p A_{i,Y_i^j,Y_{i+1}^j}^{\mathbb{Y}_i^j,\mathbb{Y}_{i+1}^j}}{\prod_{j=1}^p E \left[A_{i,Y_i^j,Y_{i+1}^j}^{\mathbb{Y}_i^j,\mathbb{Y}_{i+1}^j} \right]} \right] \right],$$

where $\mathbf{Y}^i = (Y^i, \mathbb{Y}^i)$ are independent copies of $\mathbf{Y} = (Y, \mathbb{Y})$.

Proof of Lemma 5.2. We remark the following facts:

- i) If $y \neq y'$, then $A_{i,x,y}^{\mathbf{x},\mathbf{y}} A_{i,x,y'}^{\mathbf{x},\mathbf{y}'} = 0$ almost surely. Especially, for $\{\mathbf{Y}_i^j : i = 0, \dots, n\}$ and $\{\mathbf{Y}_i^{j'} : i = 0, \dots, n\}$, if there exists an i such that $\mathbf{Y}_i^j = \mathbf{Y}_i^{j'}$ and $Y_{i+1}^j \neq Y_{i+1}^{j'}$, then

$$\prod_{i=0}^{n-1} E \left[\frac{\prod_{j=1}^p A_{i,Y_i^j,Y_{i+1}^j}^{\mathbb{Y}_i^j,\mathbb{Y}_{i+1}^j}}{\prod_{j=1}^p E \left[A_{i,Y_i^j,Y_{i+1}^j}^{\mathbb{Y}_i^j,\mathbb{Y}_{i+1}^j} \right]} \right] = 0,$$

almost surely.

- ii) If $\mathbf{y}/\mathbf{x} = k$, $\mathbf{y}'/\mathbf{x} = \ell$, and $k \leq \ell$, then $A_{i,x,y}^{\mathbf{x},\mathbf{y}} A_{i,x,y'}^{\mathbf{x},\mathbf{y}'} = A_{i,x,y}^{\mathbf{x},\mathbf{y}}$ almost surely.
- iii) If $\{\mathbf{x}^j : j = 1, \dots, p\}$ are different from each other and $\mathbf{y}^j/\mathbf{x}^j = k_j$, then $E \left[\prod_{j=1}^p A_{i,x^j,y^j}^{\mathbf{x}^j,\mathbf{y}^j} \right] = \left(\frac{1}{2d} \right)^p \sum_{s_1 \geq k_1} \dots \sum_{s_p \geq k_p} E \left[\prod_{j=1}^p q_{i,x^j}(s_j) \right]$.

Thus, the possible cases are the followings:

$$E \left[E_{\mathbf{Y}^1 \dots \mathbf{Y}^p} \left[\frac{\prod_{j=1}^p A_{i,Y_i^j,Y_{i+1}^j}^{\mathbb{Y}_i^j,\mathbb{Y}_{i+1}^j}}{\prod_{j=1}^p E \left[A_{i,Y_i^j,Y_{i+1}^j}^{\mathbb{Y}_i^j,\mathbb{Y}_{i+1}^j} \right]} \middle| Y_i^j = x^j, \mathbb{Y}_i^j = \mathbf{x}^j \text{ for } j = 1, \dots, p \right] \right]$$

$$= \begin{cases} 1 & x^j \text{ are different from each others,} \\ E \left[\prod_{j=1}^p m_{i,y^j}^{(1)} \right] & \text{if } \mathbf{x}^j \text{ are different from each others,} \\ (A), & \end{cases}$$

where (A) is the other case described as below.

We divide the set $\{1, \dots, p\}$ into the disjoint union such that

$$\{1, \dots, p\} = \prod_{k=j_1}^{j_p} I_k, \quad (5.4)$$

where $I_k = \{j \in \{1, \dots, p\} : \mathbb{X}^j = \mathbb{X}^k\}$ and j_1, \dots, j_p is the set of index of equivalence class I_k . For $\mathbb{Y}^j / \mathbb{X}^j = k_j$, we set $K_{j_\ell} = \min\{k_j : j \in I_{j_\ell}\}$. Then, we have that

$$\begin{aligned} E \left[E_{\mathbf{Y}^1 \dots \mathbf{Y}^p} \left[\frac{\prod_{j=1}^p A_{i, Y_i^j, Y_{i+1}^j}^{\mathbb{Y}_i^j, \mathbb{Y}_{i+1}^j}}{\prod_{j=1}^p E \left[A_{i, Y_i^j, Y_{i+1}^j}^{\mathbb{Y}_i^j, \mathbb{Y}_{i+1}^j} \right]} \mathbf{1} \left\{ \mathbb{Y}_{i+1}^j = \mathbb{Y}^j \text{ for } j = 1, \dots, p \right\} \middle| Y_i^j = x^j, \mathbb{Y}_i^j = \mathbb{X}^j \text{ for } j = 1, \dots, p \right] \right] \\ = E \left[\prod_{\ell=j_1}^{j_p} \left(\sum_{k \geq K_\ell} q_{i, x^\ell}(k) \right) \right]. \end{aligned}$$

By the above argument, we find that $\mathbf{Y}^1, \dots, \mathbf{Y}^P$ evolves according the following steps:

- i) First, the set process $\{S(m) : m = 0, \dots, n\}$ starts from the set $I^{(0)} = \{1, \dots, p\}$ until time $i^{(1)}$, and then it splits into some sets $I^{(1,1)}, \dots, I^{(1,k^{(1)})}$. ($i^{(1)}$ is the last time when \mathbf{Y}_i^j coincide and $I^{(1,1)}, \dots, I^{(1,k^{(1)})}$ are the equivalent class defined in (5.4) for $\mathbb{Y}_{i^{(1)}+1}^j$).
- ii) When the set process $S(m) = \{I^{(\ell,1)}, \dots, I^{(\ell,k^{(\ell)})}\}$, it jumps to the new sets $\{I^{(\ell+1,1)}, \dots, I^{(\ell+1,k^{(\ell+1)})}\}$ where each $I^{(\ell+1,r)}$ is a partition of some set of $I^{(\ell,1)}, \dots, I^{(\ell,k^{(\ell)})}$ at some time $i^{(\ell+1)}$. ($\mathbf{Y}^{(j)}$, $j \in I^{(\ell,s)}$ for each $s = 1, \dots, k^{(\ell)}$ coincides until time $i^{(\ell+1)}$ and $\mathbb{Y}_{i^{(\ell+1)}+1}^j \neq \mathbb{Y}_{i^{(\ell+1)}+1}^{j'}$ for some $j, j' \in I^{(\ell,k)}$ for some k).
- iii) If $S(m) = \{\{1\}, \dots, \{p\}\}$, then $S(m) = S(m')$ for $m' \geq m$.

First, we remark that the combination of $i^{(1)}, \dots, i^{(p-1)}$ (it may stops for less steps) are at most n^p -th order. Also,

$$\begin{aligned} E \left[E_{\mathbf{Y}^1 \dots \mathbf{Y}^p, S} \left[\frac{\prod_{j=1}^p A_{i, Y_i^j, Y_{i+1}^j}^{\mathbb{Y}_i^j, \mathbb{Y}_{i+1}^j}}{\prod_{j=1}^p E \left[A_{i, Y_i^j, Y_{i+1}^j}^{\mathbb{Y}_i^j, \mathbb{Y}_{i+1}^j} \right]} \mathbf{1} \left\{ i^{(\ell)} = i \right\} \middle| Y_i^j = x^j, \mathbb{Y}_i^j = \mathbb{X}^j \text{ for } j = 1, \dots, p \right] \right] \\ \leq C(p)K, \end{aligned}$$

and

$$\begin{aligned} E \left[E_{\mathbf{Y}^1 \dots \mathbf{Y}^p, S} \left[\frac{\prod_{j=1}^p A_{i, Y_i^j, Y_{i+1}^j}^{\mathbb{Y}_i^j, \mathbb{Y}_{i+1}^j}}{\prod_{j=1}^p E \left[A_{i, Y_i^j, Y_{i+1}^j}^{\mathbb{Y}_i^j, \mathbb{Y}_{i+1}^j} \right]} \mathbf{1} \left\{ i^{(\ell)} \neq i, \text{ for } \ell = 1, \dots, p \right\} \middle| Y_i^j = x^j, \mathbb{Y}_i^j = \mathbb{X}^j \text{ for } j = 1, \dots, p \right] \right] \\ \leq \prod_{k \in \mathcal{K}} E \left[(m_{i, x^k})^{\#\{j: x^j = x^k\}} \right] \leq \prod_{k \in \mathcal{K}} E \left[(m_{i, x^k})^p \right]^{\#\{j: x^j = x^k\}/p} \leq E \left[(m_{i, x^k})^p \right] \mathbf{1}_{\{x^j = x^k, \text{ for some } j \neq k\}}, \end{aligned}$$

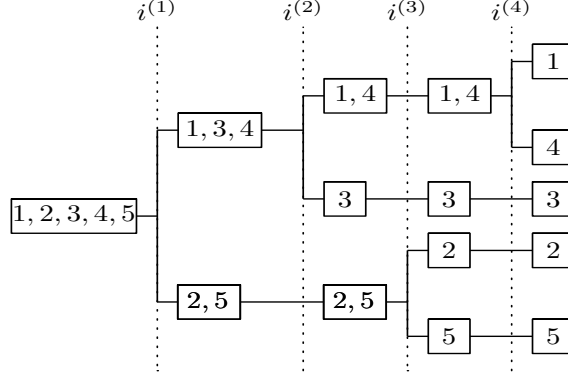


Figure 2: When $p = 5$, $I^{(0)} = \{1, 2, 3, 4, 5\}$. In this figure, $I^{(1,1)} = \{1, 3, 4\}$, $I^{(1,2)} = \{2, 5\}$, $I^{(2,1)} = \{1, 4\}$, $I^{(2,2)} = \{3\}$, and $I^{(2,3)} = \{2, 5\}$.

where \mathcal{K} be the set of index for equivalence class $\{j : x^j = x^k\}$.

Thus, we have that

$$E[B_n^P] \leq C(p, K)n^{(p-1)}E_{\mathbf{Y}^1 \dots \mathbf{Y}^p} \left[E[(m_{n,x})^p]^{\#\{i \leq n : Y_i^j = Y_i^{j'} \text{ for } j \neq j' \in \{1, \dots, p\}\}} \right].$$

The latter part of Lemma 5.2 can be proved by the same argument. \square

Corollary 5.3. *Under the same assumption in Lemma 5.2,*

$$E \left[\prod_{j=1}^q \prod_{i=1}^{p_j} B_{n,x(j,i)}^{(j)} \right] \leq C(\mathbf{p}, K)n^{(\sum_{j=1}^q p_j - q)} E_{(\mathbf{Y}^{j,i})} \left[E[(m_{0,0})^{\sum_{j=1}^q p_j}]^{\#\left\{ \begin{array}{l} k \leq n : Y_k^{j_1, i_1} = Y_k^{j_2, i_2}, \text{ for} \\ (j_1, i_1) \neq (j_2, i_2) \in \{(j, i) : j=1, \dots, q, i=1, \dots, p_j\} \end{array} \right\}} : Y_n^{(j,i)} = x_{j,i} \right],$$

where $B_{n,x}^{(j)}$ is the number of particles from initial particle j at site x at time n .

Proof. If we regard $i^{(1)} = -1$ and $S(0) = \{\{1, \dots, p_1\}, \dots, \{\sum_{j=1}^{q-1} p_j + 1, \dots, \sum_{j=1}^q p_j\}\}$, then $S(m)$ stops at $\{\{1\}, \dots, \{\sum_{j=1}^q p_j\}\}$ at most $\sum_{j=1}^q p_j - q$ jumps. \square

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